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Matching Mechanisms for Refugee Resettlement*

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Abstract

Tens of thousands of refugees are permanently resettled from refugee camps to hosting countries every year. In the past, placement of refugees was essentially ad hoc, but more recently resettlement agencies have been trying to place refugees systematically in order to improve their outcomes. Yet, even at present, refugee resettlement processes account for neither the priorities of hosting communities nor the preferences of refugees themselves. Building on models from two-sided matching theory, we introduce a new framework for *matching with multidimensional constraints* that models refugee families' needs for multiple units of different services, as well as the service capacities of localities. We propose four refugee resettlement mechanisms and two solution concepts that can be used in refugee resettlement matching under various institutional and informational constraints. Our mechanisms can improve match efficiency, respect priorities of localities, and incentivize refugees to report where they would prefer to settle.

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1 Introduction

In 2018, 70.8 million people were displaced by conflict around the world—the highest level ever recorded (UNHCR, 2019). Over 20 million of these forcibly displaced people are deemed to be *refugees* under the mandate of the United Nations High Commissioner for Refugees (UNHCR). The UNHCR estimates that, in 2020, around 1.44 million refugees will not be able return to their home countries safely in the future (UNHCR, 2019). The UNHCR deems these refugees eligible for *resettlement* in states that agree to give them permanent residence and a route to citizenship. Refugees eligible for resettlement are some of the most vulnerable refugees in the world, including children, survivors of torture and persecution, as well as women and girls at risk of violence (UNHCR, 2019). While the number of refugees eligible for resettlement has roughly doubled since 2012, the number of resettled refugees has been falling in recent years. Resettlement places are provided by countries voluntarily—and the largest hosts are the United States, Canada, the United Kingdom, France, Sweden, and Australia. Since the Second World War, the U.S. has admitted the majority of resettled refugees. For example, between 2012 and 2018, the U.S. admitted an average of 46,000 refugees every year.

Yet little attention has been paid to the process that determines where in the hosting country refugees are resettled. Most countries have historically treated refugee resettlement as a purely administrative issue, and as such have not developed systematic and transparent resettlement procedures. There is, however, ample empirical evidence that the local communities (*localities*) where refugees are initially resettled matter a great deal for refugees' education, job prospects, and earnings (Åslund and Rooth, 2007; Åslund and Fredriksson, 2009; Åslund et al., 2010, 2011; Damm, 2014; Feywerda and Gest, 2016; Bansak et al., 2018; Martén et al., 2019). In particular, the initial match matters for refugees' lifetime outcomes because most refugees do not move from the localities to which they are resettled for many years.

As of May 2018, the Hebrew Immigrant Aid Society (HIAS), a resettlement agency operating in the U.S has been systematically matching refugees according to their likelihood of gaining employment while taking various integration constraints into account (Trapp et al., 2018). Such systematic matching has the potential to increase the short-term employment of resettled refugees from 30 percent to over 40 percent while ensuring that all integration needs are met (Bansak et al., 2018; Trapp et al., 2018).

HIAS's pioneering matching software, Annie™ MOORE is based on a model of matching with multidimensional constraints that we introduced in our original working paper (Delacrétaz et al., 2016; Trapp et al., 2018).[1] Although Annie™ MOORE uses our model, the system does not account either for refugees' personal preferences over localities or for the priorities of the localities themselves.

Refugees' preferences matter because refugees have private information about their own skills and abilities, which can affect the refugee-locality match quality—and which cannot be directly observed by government authorities. Meanwhile, respecting priorities and hosting capacities of localities can improve integration-relevant outcomes of refugees, ensure the best use of local resources, and encourage localities to continue participating in resettlement by building community support.

In this paper, we consider how refugees' preferences and localities' priorities can be incorporated into refugee resettlement processes, such as the one used by HIAS. We introduce and analyze several matching market design approaches that balance competing objectives of refugee welfare, incentives, and respect for localities' priorities. Our analysis draws upon classic matching models from contexts such as public school choice and housing allocation. In a school choice model, each school has a number of school seats and any one student takes up exactly one school seat.[2] By contrast, in refugee resettlement, localities typically have a maximum number of refugees they can resettle but families must be kept together. Therefore, families have different *sizes*: larger families (e.g., a couple with four children) take up more places than smaller ones (e.g., a single individual). In addition, families require a certain number of units of different public services, such as school places, slots in language classes, specialized medical or social support, and employment training programs. Thus, there are also explicit *multidimensional constraints* that limit the central authority's ability to allocate refugees to localities simply on the basis of numbers of individual refugees. Family sizes and multidimensional constraints render most standard matching mechanisms for allocation of objects, houses, or school seats insufficient for refugee resettlement.

Our theoretical contribution. We start with the case in which the resettlement agency focuses only on the preferences of refugee families. We show that the Multidimensional Top Trading Cycles (MTTC) algorithm—a slight modification of the classical Top Trading Cycles (TTC) algorithm of Shapley and Scarf (1974)—allows us to incorporate multidimensional capacities of localities and obtain a Pareto-efficient mechanism in which refugee families do

[1] In this case our model can be analyzed as an integer program called the *multiple multidimensional knapsack problem* (Delacrétaz et al., 2016; Trapp et al., 2018).

[2] Students have heterogeneous preferences over schools and schools have priorities over students (having a sibling or living in the neighborhood typically gives students a higher priority). The social planner's objective is to elicit truthful preferences over schools from students (schools are assumed to be non-strategic and school seats are treated as objects) and to deliver a non-wasteful matching of students to schools in which no student envies another student's seat.

not have any incentive to misreport their preferences (Proposition 1). In practice, however, resettlement agencies already have existing allocation processes so we consider how to incorporate preference information into a setting with a baseline allocation, i.e, an *endowment*. For example, HIAS might start with the employment-maximizing outcome from Annie™ MOORE and then give refugees an option to submit preferences in order to improve their matches. A matching is then *individually rational* if every family is matched to a locality it weakly prefers to its endowment. In this case, because of multidimensional constraints, a Pareto-efficient and individually rational matching cannot be achieved by only using the trading cycles that arise in the MTTC algorithm: It might be necessary to swap sets of families in order to guarantee feasible Pareto improvements. Indeed, it turns out that there is no strategy-proof mechanism that be can guaranteed to find even a single Pareto improvement upon an endowment that is not Pareto-efficient (Proposition 2). We therefore relax Pareto efficiency by considering *Pareto-improving chains* in which swaps occur at the level of families. We define a matching to be *chain-efficient* if it cannot be improved by carrying out any Pareto-improving chain. We show that there does not exist any chain-efficient and strategy-proof mechanism (Theorem 1). In fact, a strategy-proof mechanism that Pareto improves upon an endowment that is not chain-efficient is not guaranteed to exist when there is more than one service (Theorem 2). In order to Pareto improve upon an endowment whenever possible, we introduce an algorithm, called Multidimensional Top Trading Cycles with Endowment (MTTCE), which generalizes the MTTC algorithm. (Without an endowment, both algorithms are equivalent (Proposition 3).) The MTTCE mechanism is strategy-proof and can potentially Pareto improve upon the endowment by carrying out Pareto-improving chains (Theorem 3). If there is only one service and larger families have a higher priority, then the MTTCE algorithm is guaranteed to Pareto improve upon the endowment in a strategy-proof way (Theorem 4). The single-service case is, in fact, relevant in practice as HIAS currently only uses only one service (namely, family size) for Annie™ MOORE (Trapp et al., 2018).

When priorities of localities also need to be taken into account, new trade-offs arise. In particular, stable matchings may not exist and determining whether a stable matching exists (or finding a stable matching when one exists) is a computationally intractable problem (McDermid and Manlove, 2010; Biró and McDermid, 2014).[3] In our model, stable outcomes only exist under fairly strong conditions (e.g., if larger families have a higher priority and require more units of all services than smaller families; see Proposition 4). To address the

[3] We use the term *stable* in the sense of *pairwise stability* following the two-sided matching literature (Gale and Shapley, 1962). In matching markets with priorities, a stable matching is sometimes said to *eliminate justified envy* (Abdulkadiroğlu and Sönmez, 2003) or to be non-wasteful and *fair* (Balinski and Sönmez, 1999).

non-existence and computational shortcomings of stable matchings, we introduce an alternative solution concept called *weak envy-freeness*, which is based on envy-freeness (Sotomayor, 1996; Wu and Roth, 2018; Kamada and Kojima, 2018), but is less demanding in a matching markets with multidimensional constraints. Envy-freeness eliminates envy, that is, it rules out matchings in which a family f is matched to locality ℓ but prefers locality ℓ' which hosts a family f' with a lower-priority than f at ℓ'. Weak envy-freeness is less stringent than envy-freeness as it rules out only that a family f' at ℓ' is envied by f whenever f' does not require any of the services over-demanded by higher-priority (than f' at ℓ') families that prefer to be at ℓ'. We show that a family-optimal weakly envy-free matching exists and can be found via a modification of the classical Deferred Acceptance (DA) algorithm (Gale and Shapley, 1962), which we call the Cascading Multidimensional Deferred Acceptance (CMDA) algorithm (Theorem 5). However, unlike in contexts such as school choice, this modification of the DA algorithm is manipulable because localities' choice functions (induced by the priorities and constraints) do not satisfy the cardinal monotonicity condition (Alkan, 2002; Hatfield and Milgrom, 2005). In fact, there is no family-optimal weakly envy-free and strategy-proof mechanism (Proposition 7), implying a trade-off between incentives and efficiency when the designer requires weak envy-freeness.[4] We therefore develop a strategy-proof and weakly envy-free mechanism, called the Threshold Multidimensional Deferred Acceptance (TMDA) algorithm (Theorem 6).

Relationship to prior work. Matching markets for refugee resettlement were first proposed by Moraga and Rapoport (2014) as a part of a system of international refugee quota trading (Schuck, 1997). In the international context of matching refugees to countries, however, the refugee matching market is "thick"—any country can be expected to host any family up to its capacity—and can be reasonably modeled as a standard school choice problem (Abdulkadiroğlu and Sönmez, 2003; Jones and Teytelboym, 2017a). Jones and Teytelboym (2017b) informally introduced the idea of refugee resettlement matching in the national context and pointed out the multidimensional constraints and the thinness of matching markets that arise on the local level. Andersson and Ehlers (2019) examine a market for allocating private housing to refugees in which landlords have preferences over the sizes of refugee families and over the native languages refugees speak.

Our work draws upon and contributes to the applied literature on the design and implementation of complex matching mechanisms. The most famous example is the National Resident Matching Program (NRMP), in which residents may apply to jobs as couples (Roth

[4]We leave the study of mechanisms that satisfy weaker conditions on truth-telling incentives, such as Bayesian incentive compatibility (Ehlers, 2008), regret-freeness (Fernandez, 2017), or partial strategy-proofness (Mennle and Seuken, 2014), for future work.

and Peranson, 1999; Klaus and Klijn, 2005; Klaus et al., 2007; Haake and Klaus, 2009). In this market, as in ours, stable outcomes may not exist. There are a number of algorithms that can find stable matchings in the couples model whenever they exist (Echenique and Yenmez, 2007; Kojima, 2015) or find approximate solutions (Nguyen and Vohra, 2018). However, the structure of our problem is different to the matching with couples problem as the barriers to stability in our context arise from the constraints on the locality (hospital) side, rather than from the family (doctor) side (as in the couples problem). While stable matchings are computationally hard to find in our model (even when they exist), there are various weaker stability concepts that are computationally tractable (Aziz et al., 2018). Stable outcomes also do not exist in general in the market for trainee teachers in Slovakia and Czechia, where teachers are expected to teach two out of three subjects and schools have capacities for each subject (Cechlárová et al., 2015). Another difficult case for market design has been matching with minimum quotas, in which stable outcomes also typically do not exist (Goto et al., 2014; Fragiadakis et al., 2016). In similar spirit to this paper, Kamada and Kojima (2018) consider many-to-one matching markets under general constraints. While their model is more general than ours, their results are independent of ours, and they focus on the structure of constraints that allow for an existence of a feasible, individually rational, and envy-free matchings. Milgrom and Segal (2019) study a dynamic auction with multidimensional constraints, but they focus mainly on the properties of the classic deferred acceptance algorithm. Finally, Nguyen et al. (2019) look for near-feasible group-stable outcomes in a version our model in which localities have cardinal preferences.

In some matching market design settings, such as school choice in New Orleans or housing allocation, stability is considered secondary to efficiency. In such cases, the Top Trading Cycles algorithm (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003) or its modifications (Pápai, 2000; Dur and Ünver, 2015) are used instead of stable and strategy-proof mechanisms. Pycia and Ünver (2017) show that in settings where agents have single-unit demands over objects, all Pareto-efficient mechanisms that cannot be manipulated by a group of agents can be represented in terms of a general class of Trading Cycles mechanisms. Pápai (2003) and Pápai (2007) analyze the difficulties of exchange with endowments and multiple goods. In our setting, families are only endowed with at most one locality; however, efficient and strategy-proof mechanisms are similarly hard to find.

Organization of the paper. The remainder of the paper is organized as follows. In Section 2, we describe the institutional context of the refugee resettlement in the U.S.. We state the formal model in Section 3. In Section 4, we explain how two variations on the Top Trading Cycles algorithm can fully incorporate preferences of refugees. In Section 5, we propose solutions for the case where refugee preferences need to be balanced with priorities of the localities. Section 7 is a conclusion. All proofs are in the Appendix. The Online Appendix provides additional results on the Threshold Multidimensional Deferred Acceptance Algorithm (Online Appendix A), examples of how different algorithms work (Online Appendix B), and relationships between our model and previous models (Online Appendix C).

2 Institutional context

The Vietnam War and the evacuation of over 130,000 Vietnamese, Cambodian, and Laotian refugees to the U.S. in 1975 precipitated the Refugee Act of 1980, which created the federal Office for Refugee Resettlement. This Act standardized the resettlement process, set flexible annual quotas, and fixed funding for resettlement. Since then around three million refugees have been resettled to the U.S., mainly from southeast Asia and the former Soviet Union. Annual resettlement numbers fluctuate considerably, partly because they can be altered by executive action in response to crises and political will. In the past decade, around 70,000 refugees arrived annually to the U.S.; this rose to almost 78,761 in 2016. In September 2016, President Obama had committed to resettling at least 110,000 refugees in 2017; however, in January 2017 President Trump reduced the quota to 50,000 (although only 24,559 refugees were eventually resettled to the U.S. in 2017 and 17,112 in 2018).

Refugees can apply for the U.S. resettlement program directly or be referred by the UNHCR (often while living in a refugee camp). The refugee resettlement program is managed by the U.S. Refugee Admissions Program (USRAP) which, alongside the UNHCR and the International Organization for Migration, identifies refugees, conducts security checks on all family members, and arranges travel;[5] This process can take 18 to 24 months. Once a family has passed the security checks, it proceeds to medical checks and cultural orientation. Then the case is handed over to one of the nine U.S. Resettlement Agencies, which are responsible for matching the refugee family to a local community.[6] Resettlement agencies resettle refugees all over the U.S., although the majority are initially placed in California, Florida, New York State, and Texas.

[5] USRAP consists of the Bureau of Population, Refugees and Migration (PRM) of the U.S. Department of State, the U.S. Citizenship and Immigration Services (USCIS) of the U.S. Department of Homeland Security, and the Office of Refugee Resettlement (ORR) of the U.S. Department of Health and Human Services (HHS)

[6] These resettlement agencies, also known as "voluntary agencies", are: Church World Service (CWS), Ethiopian Community Development Council (ECDC), Episcopal Migration Ministries (EMM), Hebrew Immigrant Aid Society (HIAS), International Rescue Committee (IRC), U.S. Committee for Refugees and Immigrants (USCRI), Lutheran Immigration and Refugee Services (LIRS), U.S. Conference of Catholic Bishops (USCCB), World Relief Corporation (WR).

Refugees are allowed to list family members who live in the U.S., in which case they are almost certain to be reunited with them. But beyond that, resettlement agencies do not collect information about refugees' preferences over initial placements, and instead must make informed guesses about where refugees would fare well.

Resettlement agencies establish their own links to local communities, which we refer to as *localities* throughout the paper, that are willing to host refugees.[7] Every year, agencies review the capacities of their localities to host refugees. Hosting commitments of localities run for the duration of the fiscal year (from October to October). Localities express a variety of hosting constraints to their agencies. The key constraint is the total number of refugees they are able or willing to host. The Department of State approves the quotas for every locality, which can be as high as 200 or as low as 20 refugees per year. Localities cannot exceed the annual quotas. There might also be capacity constraints on services, such as school places or employment training, as well as on the number of refugees from certain countries or regions. Other (binary) constraints might include whether the community can support single-parent families or disabled refugees. Agencies assign refugees to localities roughly every fortnight. In order to balance the resettlement load, the fortnightly quota for each locality is set proportionally to that locality's annual quota and is treated as a hard constraint (Trapp et al., 2018). Localities do not currently fix priorities over specific types of refugees beyond the constraints they express to agencies. Most localities commit to supporting refugees for their first year of resettlement, after which the refugees are more or less on their own.[8]

2.1 Introducing preferences and priorities into refugee resettlement

In 2018, HIAS became the first resettlement agency to adopt a matching system that attempts to maximize short-run refugee employment while meeting the constraints of localities. The system predicts employment likelihoods across refugee-locality matches using observable historical data and suggests matchings that aim to maximize the employment objective. While matching families and localities based on observable characteristics constitutes a significant improvement over random allocation or ad hoc procedures (Bansak et al., 2018; Trapp et al., 2018), there are still reasons to incorporate refugees' preferences and localities'

[7] Resettlement agencies also coordinate the entire arrival process with the locality and ensure that housing and initial support facilities (e.g., airport pickup, pocket money, first meals) are ready when the refugee family arrives.

[8] The federal government provides very limited support with a fixed grant of $1,125 per refugee to cover the first 90 days of resettlement.

priorities into the matching process.[9]

Refugees are likely to hold private information about their preferences that can affect the quality of matches. As Mark Hetfield, the CEO of HIAS said:

> "Many Somali refugees initially settled around the country subsequently migrated to Lewiston, Maine. Lewiston has a weak economy but an established Somali community. Consequently, efforts to resettle these refugees elsewhere in the U.S. were less effective than they could have been. Their preferences should have been taken into account from the start" (see Roth (2015)).

Indeed, refugees' private information might even be pertinent to maximizing observable outcomes, such as employment. As Hedfield points out, taking preferences into account may also help prevent internal migration—the movement of refugees away from their assigned localities soon after arrival—which localities want to avoid because they make substantial upfront investments in hosting refugees. On the other side of the match, resettlement agencies also miss information by not eliciting priorities from localities. For example, a locality cannot currently signal that it would prefer to resettle larger or smaller families, families from particular ethnic or linguistic groups, or families in which members have certain medical conditions.[10]

Another reason for eliciting and responding to preferences is ethical. Refugees ought to be afforded the same agency as other citizens and be given some choice during one of the most consequential moments in their lives. It is arguably inconsistent to talk about what is "good" for refugees without seriously considering their preferences. Meanwhile, respecting priorities of localities can serve as an important fairness criterion for refugee families and give communities a sense of control over the resettlement process—which might corral goodwill and incentivize localities' long-term participation.

3 Model

There is a finite set of refugee *families* $f \in F$ and a finite set of *localities* $\ell \in L$. We assume that there exists a *null* locality $\emptyset \in L$ that represents being unmatched.

Families require multiple units of different *services* from a finite set S. Let $\nu_s^f \in \mathbb{Z}_{\geq 0}$ denote the total number of units of service $s \in S$ needed by family f. We say that needs are

[9] Logistically, there is no obvious impediment to the collection of preferences and priorities. Resettlement agencies have time to elicit families' preferences over localities after those families' applications have been approved and before departure. Localities can be consulted about their priorities at the beginning of each fiscal year.

[10] A resettlement agency might put reasonable restriction on what criteria can be used for priority rankings. For example, a locality could be prevented from ranking refugees based on race or religion.

monotonic if for any two families f and f' and any two services s and s', $\nu_s^f > \nu_s^{f'}$ implies $\nu_{s'}^f \geq \nu_{s'}^{f'}$. When $|S| = 1$, needs are always monotonic; however, when $|S| > 1$, the needs monotonicity condition states that whenever f requires more units of one service than f', then f also requires weakly more units of all other services.

Let $\kappa_s^\ell \in \mathbb{Z}_{\geq 0}$ denote the number of units of service s that locality ℓ can provide. We use the convention that $\kappa_s^\emptyset = +\infty$ for all services s, i.e., that the null locality has infinite capacity for every service. Our definition of services allows for quotas on total number of refugees, capacities on public services (e.g., school places), ability to satisfy specific needs (e.g., support for particular medical conditions), or even political and cultural constraints (e.g., number of refugees from a particular region).

A family f can only live in locality ℓ if ℓ can provide services to meet f's needs. Locality ℓ can *accommodate* a set of families $G \subseteq F$ if $\sum_{g \in G} \nu_s^g \leq \kappa_s^\ell$ for all $s \in S$. Locality ℓ can *accommodate* a family f *alongside* $G \subseteq F \setminus \{f\}$ if ℓ can accommodate $G \cup \{f\}$. We assume that every family can be accommodated on its own by at least one locality other than the null.

A (feasible many-to-one) *matching* is a correspondence $\mu : F \cup L \rightrightarrows F \cup L$, such that for all $f \in F$, $\ell \in L$, and $s \in S$,

(i) every family is matched to exactly one locality, i.e., $\mu(f) \in L$;

(ii) every locality is matched to a subset of families, i.e., $\mu(\ell) \subseteq F$;

(iii) a family is matched to a locality if and only if the locality is matched to the family, i.e., $\mu(f) = \ell$ if and only if $f \in \mu(\ell)$; and

(iv) every locality can accommodate all the families matched to it, i.e., $\sum_{g \in \mu(\ell)} \nu_s^g \leq \kappa_s^\ell$.

The first three conditions are standard while condition (iv) ensures that service capacities are not exceeded in any locality. Figure 1 illustrates an example of a matching in a market with multidimensional constraints. Our model generalizes a number of existing matching models with complex constraints (see Online Appendix C).

Families have *preferences* over localities. We denote by $\succ_f$ the strict ordinal preference list of family f over L, and let $\succ = (\succ_f)_{f \in F}$ be the preference profile of families. We therefore write $\ell \succ_f \ell'$ to mean that f strictly prefers ℓ to ℓ'. We write $\ell \succeq_f \ell'$ to denote that either $\ell \succ_f \ell'$ or $\ell = \ell'$. We assume that every family's least preferred option is being unmatched.

Localities have exogenously fixed *priorities* over families. We let $\rhd_\ell$ be the strict ordinal priority list of locality ℓ over families F and let $\rhd = (\rhd_\ell)_{\ell \in L}$ be the ordinal priority profile of the localities. We therefore write $f \rhd_\ell f'$ to mean that f has a higher priority than f' at ℓ.

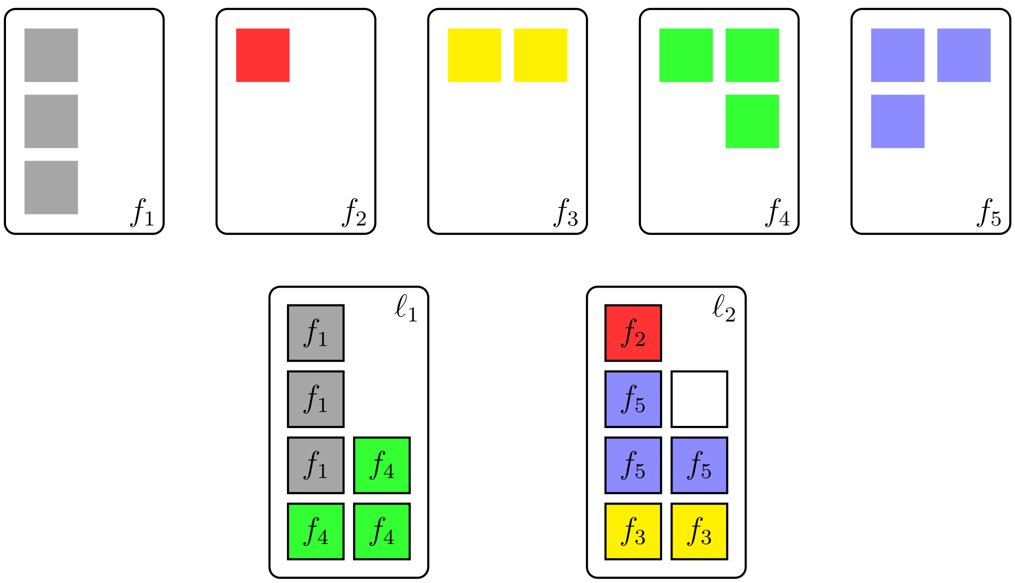

Figure 1: Matching in a market with multidimentional constraints. There are five families $f_1, \ldots, f_5$, two localities ℓ_1, ℓ_2, and two services, represented by the left and right columns. The needs of the families are $(3, 0)$, $(1, 0)$, $(1, 1)$, $(1, 2)$, and $(2, 1)$; for example, family f_1's needs are represented by 3 blocks in the first column and 0 blocks in the second column. The capacities of the localities are $(4, 2)$ and $(4, 3)$. In the matching pictured, families f_1 and f_4 are matched to locality ℓ_1 and families f_2, f_3, and f_5 are matched to locality ℓ_2. All of ℓ_1's capacity is used while ℓ_2 has one unit of the second service that remains unused.

We assume that localities prioritize families they can accommodate on their own: if ℓ can accommodate $\{f\}$ but not $\{f'\}$, then $f \rhd_\ell f'$.

A matching μ is *wasteful* if there exists a family $f \in F$ and a locality $\ell \in L$ such that $\ell \succ_f \mu(f)$ and ℓ can accommodate f alongside $\mu(\ell)$. We say that matching μ *Pareto dominates* matching μ', denoted $\mu \succ \mu'$, if $\mu(f) \succeq_f \mu'(f)$ for all $f \in F$ and $\mu(f) \succ_f \mu'(f)$ for some $f \in F$. We write $\mu \succeq \mu'$ if μ *weakly Pareto dominates* μ', that is if either $\mu \succ \mu'$ or $\mu = \mu'$. A matching μ is *Pareto-efficient* if there does not exist another matching μ' that Pareto dominates μ.

Fixing a set of families, a set of localities, a profile of priorities, as well as service needs and capacities, we define a (direct) *mechanism* as a function φ that takes as input a preference profile and outputs a matching. A mechanism φ is *strategy-proof* if for any $f \in F$ there does not exist a report of a preference list $\succ'_f$ such that

$$\varphi(\succ'_f, \succ_{-f})(f) \succ_f \varphi(\succ)(f),$$

where $\varphi(\succ)(f)$ is the locality to which f is matched under the mechanism φ and the preference profile $\succ$. Strategy-proofness requires that refugee families cannot make themselves better off by misreporting their preferences over localities, irrespective of the reports of other families.

If a mechanism always selects a matching with a certain property, we refer to the mechanism as having that property. For example, if a mechanism always selects a non-wasteful matching, we call it a *non-wasteful mechanism*.

Throughout the paper, we describe all the mechanisms as algorithms: therefore, a mechanism takes a preference profile as an input and uses instructions from its corresponding algorithm to produce a matching. We say that a family is *permanently matched* (*permanently rejected*) to a locality at some step of the algorithm for mechanism φ if by that step it has been established that the family will (not) be matched to the locality in the matching outputted by φ.

4 Targeting efficiency

In this section, we propose matching mechanisms that incorporate refugee preferences and give refugees an incentive to report their preferences over localities truthfully.

We show first that the Pareto-efficient and strategy-proof Top Trading Cycles (TTC) mechanism of Shapley and Scarf (1974) can be adapted to our setting in a natural way while preserving its properties. However, resettlement agencies already have existing processes for assigning refugees to localities—for example, HIAS matches refugees according to observable characteristics to maximize objectives such as the likelihood of employment (Bansak et al., 2018; Trapp et al., 2018). To make preference-based matching as easy as possible to integrate with existing systems, we extend our TTC-based approach to allow resettlement agencies to use their initial allocations—which we refer to as an *endowment*—as a baseline and allow refugees to express preferences in order to improve upon that baseline.

We consider mechanisms that Pareto improve upon the endowment in the sense that every refugee family is assigned to a locality that it weakly prefers to its endowment. Pareto improvements upon an endowment make it possible to use refugees' preferences to identify welfare gains over and above the assignments that would be selected based on observables. Such Pareto improvements also allow, in principle, resettlement agencies to institute distributional goals or other minimal welfare guarantees through the choice of the endowment. We show that finding Pareto improvements upon an endowment presents many challenges in our environment; even so, we obtain a strategy-proof mechanism based on the TTC algorithm that can achieve Pareto improvements.

4.1 The Multidimensional Top Trading Cycles mechanism

Our first mechanism, described in Algorithm 1, is an extension of the TTC mechanism to matching with multidimensional constraints.

In the first round, each family f *points* at its most preferred locality ℓ that can accommodate f and each locality ℓ' points at the highest-priority family f' that ℓ' can accommodate. There must be at least one (directed) cycle in the graph in which nodes are labeled with families and localities and "pointing" is represented by directed edges. Note that each family and each locality is in at most one cycle. Every family in a cycle is permanently matched to the locality at which it is pointing.

In the subsequent rounds, every locality permanently rejects every family that cannot be accommodated alongside families that are already permanently matched to that locality. Everyfamily then points at its most preferred locality from which the family has not yet been permanently rejected; every locality points at its highest-priority family that the locality has not yet permanently rejected. Once again, there is at least one cycle and in each cycle we permanently match every family to the locality at which that family is pointing. The algorithm terminates in a finite number of steps because at least one family is permanently matched in each step of the algorithm.

Proposition 1. *The MTTC mechanism is strategy-proof and Pareto-efficient.*

The Multidimensional Top Trading Cycles (MTTC) algorithm generalizes the classical

Algorithm 1: MULTIDIMENSIONAL TOP TRADING CYCLES (MTTC)

Initialize the *current matching* μ^1 such that $\mu^1(f) = \emptyset$ for all $f \in F$. No families are *permanently matched*.

Round $i \geq 1$

Starting with μ^i, every locality ℓ *permanently rejects* all families that ℓ cannot accommodate alongside $\mu^i(\ell)$.

Every family f that is not permanently matched points at its most preferred locality among those that have not permanently rejected f.

Every locality ℓ points at the highest-priority family that has not been permanently matched and that ℓ has not permanently rejected. (If no such family exists, then ℓ does not point.)

At least one cycle appears and every family and every locality is involved in at most one cycle. Update the current matching to μ^{i+1} by permanently matching every family in a cycle to the locality at which it is pointing (this could be $\emptyset$).

If all families are permanently matched, end and output μ^{i+1}. Otherwise continue to Round $i + 1$.

TTC algorithm used in school choice by taking into account multidimensional service constraints. The MTTC mechanism is Pareto-efficient because a family continues pointing at its most preferred locality until it can no longer be accommodated alongside families that are permanently matched to it. The key reason for strategy-proofness is that a family can only be permanently matched by being involved in a cycle. A family can only create such a cycle by pointing at any locality in a pointing sequence that terminates with the family; the pointing sequence continues to exist until the family is permanently matched. Therefore, every family has an incentive to continue pointing at its most preferred locality that has not permanently rejected it yet.

Note that Proposition 1 does not include any properties of the MTTC mechanism that pertain to the priorities of the localities. Indeed, the priorities are only used in the MTTC mechanism in order to determine the order in which localities point. However, strategy-proofness and efficiency of the MTTC mechanism do not depend on the pointing order of the localities.

4.2 Improving efficiency from an endowment

Consider an exogenous matching μ^E, which we refer to as the *endowment*. We say that a matching μ is *individually rational* if $\mu \succeq \mu^E$, that is if μ weakly Pareto dominates the endowment.[11] We will also refer to locality $\mu^E(f)$ as *family f's endowment* and to families $\mu^E(\ell)$ as *locality ℓ's endowment*.

In the school choice setting, i.e., when $|S| = 1$ and $\nu_s^f = 1$ for all f, the TTC mechanism finds an individually rational and Pareto-efficient matching by carrying out one cycle at a time.[12] In contrast, in a setting with multidimensional constraints, carrying out one cycle at a time may not achieve Pareto-efficiency because some Pareto improvements may require families to swap in groups. For example, two "small" families in one locality might be able to swap simultaneously with a "large" family in another locality, but none of the pairwise swaps between a "small" family and the "large" family would be feasible. We therefore limit the set of Pareto improvements that can be executed.

Definition 1. Given a matching μ, a *Pareto-improving chain* is a sequence

$$(f_1, \ell_1, f_2, \ell_2, \dots, f_n, \ell_n)$$

of distinct families and localities such that:

[11] Our meaning from individual rationality is in line with the rest of the social choice literature here as the endowment, rather than not being matched, now plays the role of the outside option.

[12] The MTTC mechanism collapses to the TTC mechanism in the school choice setting.

- $\ell_1 \succ_{f_1} \mu(f_1)$;
- for all $i = 2, \ldots, n$,
  - ℓ_{i-1} can accommodate f_{i-1} alongside $\mu(\ell_{i-1}) \setminus \{f_i\}$,
  - $\ell_i \succ_{f_i} \ell_{i-1} = \mu(f_i)$; and
- ℓ_n can accommodate f_n alongside $\mu(\ell_n) \setminus \{f_1\}$.

In any Pareto-improving chain, family f_1 moves to locality ℓ_1 which f_1 prefers to its current locality. Locality ℓ_1, in turn, must be able to accommodate f_1 alongside all families in $\mu(\ell_1)$ except for f_2 which leaves locality ℓ_1 for a more preferred locality ℓ_2. The Pareto-improving chain continues with f_3 moving from ℓ_2 to ℓ_3 and so on. The Pareto-improving chain terminates in one of two ways: either (i) the Pareto-improving chain is "open" and no family leaves the last locality, i.e., $\ell_n \neq \mu(f_1)$ or (ii) the Pareto-improving chain is "closed" and f_1 leaves the last locality, i.e., $\ell_n = \mu(f_1)$.

Definition 2. A matching is *chain-efficient* if it does not have any Pareto-improving chain.

Chain efficiency constitutes a relaxation of Pareto efficiency because it only requires the elimination of Pareto-improving chains, which form a subset of all possible Pareto improvements. In school choice, Pareto efficiency is equivalent to chain efficiency and the Pareto-efficient TTC mechanism is strategy-proof. In our setting with multidimensional constraints, however, there might be a matching that Pareto dominates a chain-efficient matching if there are groups of families that could participate in a Pareto-improving swap that is not a Pareto-improving chain. Moreover, even chain-efficient mechanisms are not strategy-proof.

Theorem 1. *There is no strategy-proof, individually rational, and chain-efficient mechanism.*

The proof of Theorem 1 requires only one service and the largest family "size" is two ($\nu_s^f \leq 2$). The intuition is that different Pareto-improving chains can interfere with each other, thereby giving families an opportunity to select into the Pareto-improving chain they prefer by manipulating their preferences. As Pareto efficiency implies chain efficiency, Theorem 1 directly implies the following result.

Corollary 1. *There is no strategy-proof, individually rational, and Pareto-efficient mechanism.*

More generally, Theorem 1 and Corollary 1 imply a trade-off between efficiency and strategy-proofness when the designer wants to Pareto improve upon an endowment. This trade-off does not exist in school choice and constitutes a direct consequence of the fact that families may require multiple units of different services. This leaves open an important question: can a strategy-proof mechanism guarantee even a single Pareto improvement upon an endowment? To formalize this idea, we say that a mechanism φ *Pareto improves* upon an endowment μ^E, if $\varphi(\succ) \succ \mu^E$, that is if the mechanism returns a matching that Pareto dominates the endowment. This definition strengthens individual rationality by ruling out the case where the mechanism returns the endowment.

Proposition 2. *There is no strategy-proof mechanism that Pareto improves upon every endowment that is not Pareto-efficient.*

By definition, a mechanism cannot Pareto improve upon a Pareto-efficient endowment. Proposition 2 implies that, even if it were possible to Pareto improve upon the endowment, one might not be able to do so without giving families an incentive to misrepresent their preferences. The intuition behind the proof of Proposition 2 is similar to the proof of Theorem 1.

As chain efficiency is less stringent than Pareto efficiency, one may still hope to be able to Pareto-improve upon endowments that are not chain-efficient in a strategy-proof way—but even this turns out to be impossible when there is more than one service.

Theorem 2. *If $|S| > 1$, there is no strategy-proof mechanism that Pareto improves upon every endowment that is not chain-efficient.*

Theorem 2 considers *any* kind of Pareto improvements, whether or not they are Pareto-improving chains. Therefore, the result directly implies that, when $|S| > 1$, it may not be possible to find any Pareto-improving chains—even if they exist—without giving families an incentive to misrepresent their preferences. This means that strategy-proofness may preclude *all* trade in matching markets with multidimensional constraints.

While Theorem 1 and Proposition 2 hold even for one service, the impossibility result in Theorem 2 relies on the failure of the needs monotonicity condition, which can only occur when there is more than one service. In the remainder of this section, we show how we can adapt the MTTC mechanism in order to overcome the impossibility result in Theorem 2 whenever families' needs are monotonic (e.g., when there is only one service).

4.3 MTTC with Endowment

We now present an extension of the MTTC mechanism which attempts to Pareto improve upon an endowment. As in the MTTC algorithm, the MTTC with Endowment (MTTCE) algorithm (Algorithm 2) looks for trading cycles and families point at localities which they

Algorithm 2: MTTC WITH ENDOWMENT (MTTCE)

Initialize the *current matching* μ^1 such that $\mu^1(f) = \mu^E(f)$ for all $f \in F$. No families are *permanently matched*.

Round $i \geq 1$

Every locality ℓ *permanently rejects* all families that ℓ cannot accommodate alongside families that are permanently matched to ℓ.

Every locality ℓ points at the highest-priority family that has not been permanently matched and that ℓ has not permanently rejected. (If no such family exists, then ℓ does not point.)

At least one cycle appears and every family and every locality is involved in at most one cycle. Label the families and localities in any such cycle $f_1 \to \ell_1 \to f_2 \to \ell_2, \ldots, f_n \to \ell_n \to f_1$.

A cycle is *feasible* if, for all $j = 1, \ldots, n$, ℓ_j can accommodate f_j alongside $\mu^i(\ell_j) \setminus \{f_{j+1}\}$ (letting $f_{n+1} = f_1$).

If one or more cycles are feasible, continue to the *Matching Stage*. Otherwise, continue to the *Rejection Stage*.

Matching Stage: Update the current matching to μ^{i+1} by matching every family in a feasible cycle to the locality at which it is pointing (this could be $\emptyset$); all these families become permanently matched.

If all families are permanently matched, end and output μ^{i+1}. Otherwise continue to Round $i + 1$.

Rejection Stage: Pick one family f (at random or according to some exogenous rule) at whom at least one locality is pointing. Permanently reject f from all localities to which f cannot be matched, i.e., ℓ permanently rejects f if ℓ cannot accommodate f alongside $\mu^i(\ell) \setminus \{f'\}$ (where f' is the family at which ℓ is pointing).

If f is permanently rejected by the locality at which f is pointing, let $\mu^{i+1} = \mu^i$ and continue to Round $i+1$. Otherwise, pick another family that has not been picked yet and repeat the Rejection Stage.

prefer to their endowment. However, the MTTCE algorithm checks whether the cycles that appear are feasible.[13] In general, endowments can cause trading cycles to be infeasible.[14] If trading cycles are feasible, we match families to the localities they are pointing at in the cycles just as in MTTC. The key step—the Rejection Stage—deals with the case when none of the cycles are feasible. In the Rejection Stage, we pick a family f at random or according to some exogenous rule among those at which at least one locality is pointing. Family f is permanently rejected by every locality ℓ where f cannot be accommodated alongside families that are currently matched to ℓ except for some family at which ℓ is pointing (if there is such a family). Since f is not necessarily rejected by all localities, the Rejection Stage leaves an opportunity for f to be involved in a feasible cycle in a subsequent round of the algorithm. It is worth noting that any feasible cycle found by the MTTCE algorithm corresponds to at least one Pareto-improving chain.[15] Therefore, the MTTCE algorithm attempts to improve upon the endowment by carrying out successive Pareto-improving chains (each of which may be "open" or "closed").

Theorem 3. *The MTTCE mechanism is strategy-proof and individually rational.*

The MTTCE mechanism preserves strategy-proofness because (i) in the Matching Stage, families point at their most preferred localities (as in MTTC), and (ii) in the Rejection Stage, the permanent rejections from localities do not depend on reported preferences.

Effectively, the MTTCE algorithm adds the Rejection Stage to each round of the MTTC algorithm in other to deal with infeasible cycles created by the endowment. The following proposition formalizes this point—when all families are endowed with the null locality, the outcomes of the MTTC and MTTCE algorithms coincide.

Proposition 3. *If $\mu^E(f) = \emptyset$ for all $f \in F$, then $\mu^{MTTC} = \mu^{MTTCE}$.*

We illustrate the MTTCE algorithm with an example in Online Appendix B.2.

As Theorem 2 shows, whenever $|S| > 1$, no strategy-proof mechanism (e.g., MTTCE) can be guaranteed to find any Pareto improvements upon an endowment that is not chain-efficient. Therefore, in general, MTTCE might simply output the endowment even if Pareto-improving chains exist.

[13]This is not particularly challenging from a computational point of view as it simply requires to verify whether each locality in the cycle can replace the family pointing at the locality by the family at which the locality points. Thus, the MTTCE algorithm works in polynomial time.

[14]If $|S| = 1$ and $v_s^f = 1$ for all $f \in F$ and all $s \in S$ all trading cycles are feasible. Moreover, without endowments all trading cycles are also feasible.

[15]In particular, a feasible cycle can be broken into multiple "open" Pareto-improving chains if some localities point at families not in their endowment.

However, so long as needs are monotonic (which includes the case where $|S| = 1$), there exist priority profiles for which the MTTCE mechanism improves upon every endowment that is not chain-efficient. We now provide a class of priority profiles which guarantees that the MTTCE mechanism can find a Pareto-improving chain.

Definition 3. Let needs be monotonic. A priority profile is *lexicographic* if, for every $f, g \in F$ and every $\ell \in L \setminus \{\emptyset\}$,

- families in $\ell's$ endowment have a higher priority, i.e., $f \in \mu^E(\ell)$ and $g \notin \mu^E(\ell)$ implies $f \rhd_\ell g$; and
- within $\ell's$ endowment, families with greater needs have a higher priority, i.e., $f, g \in \mu^E(\ell)$ and $\nu_s^f > \nu_s^g$ for some $s \in S$ imply $f \rhd_\ell g$.

Lexicographic priorities imply that each locality prioritizes all families in its endowment over those that are not and that, among the families in its endowment, the locality prioritizes families in decreasing order of needs. There are no restrictions about how each locality ranks any two families with the same needs or any two families not in its endowment. Note that lexicographic priorities are only well-defined when needs are monotonic. If needs are not monotonic, then there exist two families f and g such that $\nu_{s_1}^f > \nu_{s_1}^g$ and $\nu_{s_2}^f < \nu_{s_2}^g$. Lexicographic priorities dictate that every locality prioritize f over g *and* prioritize g over f, a contradiction.

If needs are monotonic and priorities are lexicographic, at the start of the MTTCE algorithm every locality ℓ points at the largest family in its endowment. Suppose that ℓ permanently rejects a family f that is not in its endowment. This means that f cannot be accommodated at ℓ even when the *largest* family in ℓ's endowment has been removed. Therefore, in the endowment, there does not exist any Pareto-improving chain in which f moves to ℓ. If the outcome of the MTTCE algorithm is the endowment, then f has been permanently rejected by every preferred locality. This means that the endowment had no Pareto-improving chains in which f moves to any of the preferred localities and f is not involved in any Pareto-improving chain in the endowment. By extending this argument to every family, we can see that the MTTCE algorithm only returns the endowment whenever the endowment is chain-efficient.

Theorem 4. *If needs are monotonic and priorities are lexicographic, then the MTTCE mechanism Pareto improves upon every endowment that is not chain-efficient.*

The MTTCE mechanism takes priorities as an input to determine the family to which each locality points (if any) in every round. However, properties presented in this section do not depend on priorities. Therefore, if needs are monotonic but priorities are not lexicographic, it is possible to modify the pointing order in the MTTCE algorithm by constructing the pointing order from lexicographic priorities. This does not affect the mechanism's properties—individual rationality and strategy-proofness—but ensures that the MTTCE mechanism finds a Pareto-improving chain as long as one exists. This observation implies the following corollary to Theorem 4.

Corollary 2. *If needs are monotonic, then there exists a strategy-proof mechanism that Pareto improves upon every endowment that is not chain-efficient.*

Recall that whenever there is only one service, needs are always monotonic ensuring the existence of a strategy-proof mechanism—i.e., MTTCE with an adjusted pointing order—that Pareto improves upon any endowment. Corollary 2 therefore contrasts the impossibility result in Theorem 2 by highlighting the possibility of strategy-proof Pareto-improvement upon endowments that are not chain-efficient in the case when there is only one service. Moreover, HIAS currently uses Annie™ MOORE with only one service—family size—in its resettlement processes (Trapp et al., 2018). Therefore, Corollary 2 is relevant for current practice. We leave open the question of finding the most efficient, strategy-proof, and individually rational mechanism.

5 Accounting for priorities

As we have argued in this paper, preferences of refugees are central to designing matching mechanisms for refugee resettlement. However, there are good reasons for taking priorities of localities seriously as well. Mechanisms in Section 4 do not guarantee to satisfy the priorities of localities. In this section, we offer mechanisms that respect priorities of localities in addition to the preferences of refugee families.

5.1 (Non-)existence of stable matchings

Denote by $\widehat{F}_\ell^f = \{g \in F : g \rhd_\ell f\}$ the set of families with a higher priority than family f at locality ℓ. A common solution concept for balancing preferences and priorities is (pairwise) *stability* (Roth, 1984a; Abdulkadiroğlu and Sönmez, 2003).

Definition 4. A matching μ is *stable* if there is no $f \in F$ and $\ell \in L$ such that

(i) f prefers ℓ to its current match, i.e., $\ell \succ_f \mu(f)$; and

(ii) ℓ can accommodate f alongside $\widehat{F}_\ell^f \cap \mu(\ell)$, i.e., all families matched to ℓ with a higher priority than f.

In words, family f and locality ℓ are a *blocking pair* in a matching μ if f prefers ℓ to its current match and it is possible to accommodate f in ℓ without removing any higher-priority family. A matching is stable if it does not have any blocking pairs. Our definition extends the concept of stability to a setting with multidimensional constraints. Our definition is in line with the way stability is defined in similar models (see, e.g., McDermid and Manlove (2010); Biró and McDermid (2014); Delacrétaz (2019)). If $|S| = 1$ and $\nu^f_{s_1} = 1$ for all $f \in F$, Definition 4 collapses to the "elimination of justified envy" used in school choice and other object allocation settings (Abdulkadiroğlu and Sönmez, 2003).

While in school choice models stable matchings always exist, in a model with multidimensional constraints they do not.[16] In fact, determining whether a stable matching exists in our model is a computationally intractable problem (McDermid and Manlove, 2010): The running time of an algorithm that can be guaranteed to find a stable matching or proves that none exists increases exponentially with the problem size.[17] This can be an impediment to practical applications in large matching markets.[18]

However, stable outcomes are guaranteed to exist in a special case of our model where needs are monotonic and the priorities of localities are sufficiently similar.

Definition 5. A priority profile $\rhd$ is *aligned* if for any $f, g \in F$ such that $\nu^f \neq \nu^g$ and any $\ell, \ell' \in L \setminus \{\emptyset\}$, $f \rhd_\ell g$ if and only if $f \rhd_{\ell'} g$.

The aligned priorities condition generalizes the second part in the definition of lexicographic priorities (Definition 3). Under the aligned priorities condition, any two families with different needs are ranked identically by all localities, but families with the same needs can be ranked arbitrarily. The case of identical priorities is therefore a special case of the aligned priorities condition. If needs are monotonic, aligned priorities also include the case where all localities give a higher priority to larger families and, symmetrically, the case where all localities give a higher priority to smaller families.

Proposition 4. *If needs are monotonic and the priority profile is aligned, then a stable matching exists.*

The monotonicity of needs and alignment of the priority profile ensures that the set of families can be partitioned into $\{F_1, F_2, \ldots, F_n\}$ such that for any $i = 1, \ldots, n$, all families

[16] In Online Appendix B.1, we present an example of a market in which a stable matching does not exist.

[17] The decision problem of the existence of stable matchings in our setting is NP-complete, meaning there is no known efficient (e.g., with polynomial running time in the number of families or localities) method of solving it. Finding a stable matching, like solving the multiple multidimensional knapsack problem, is therefore NP-hard, i.e., as hard as the hardest computational problems. Verifying whether a particular matching is stable only involves checking all possible blocking pairs which is a simple computational problem.

[18] Delacrétaz et al. (2016) present an algorithm to find a stable matching whenever one exists.

in F_i have the same needs and for all $j < i$, all families in F_j have a higher priority for all localities than all families in F_i. A stable matching then can be obtained in polynomial time by running sequentially the (family-proposing) Deferred Acceptance algorithm for each subset, starting with F_1. In a school choice setting, all families have the same needs ($\nu^f = 1$ for all $f \in F$) and priority profile is (trivially) aligned; therefore the existence of stable matchings follows immediately from Proposition 4. As in school choice, stable matchings under needs monotonicty and priority alignment can be found in polynomial time in our model.

5.2 Envy-free matchings

In the refugee resettlement context, priority alignment might be a strong assumption. In general, we should expect localities' priorities to be heterogeneous. For example, even if all localities use the same objective function, such as employment, refugee families' likelihoods of employment might vary substantially across localities. The possible non-existence of stable matchings and the computational challenges involved in finding them motivates us to consider alternative solution concepts. The key issue we will face is how to trade off respect for priorities against tolerating wasted capacity. Indeed, as Delacrétaz (2019) shows: if waste must be eliminated, then there may exist a blocking pair (f, ℓ) where an arbitrarily large number of units of ℓ are assigned to families with a lower priority than f. In the context of refugee resettlement, locality goodwill might be important; therefore respecting priorities would often take center stage. Furthermore, the underuse of service capacities may well be tolerable. Refugees arrive to many localities regularly and many services, such as school places and employment training programs, are durable and unlikely to disappear if they are not immediately used. Any unused service capacities can simply be used for the next cohort of resettled refugee families. In the refugee resettlement context, therefore, it seems reasonable to consider matching mechanisms that tilt the balance somewhat in favor of respecting priorities rather than eliminating waste.

We now introduce *envy-free* matchings, which respect the priorities of localities, but introduce possible underuse in service capacities.

Definition 6. Given a matching μ, $f \in F$ *envies* family $f' \neq f$ if

(i) f prefers f''s locality to its current match, i.e., $\mu(f') \succ_f \mu(f)$; and

(ii) f has a higher priority at f''s locality, i.e., $f \rhd_{\mu(f')} f'$.

Definition 7. A matching μ is *envy-free* if, under μ, no family envies another family.

Envy-freeness ensures that priorities are fully respected (as in a stable matching), but it tolerates waste.[19] In our setting, waste can occur when a "small" family f prefers locality ℓ to its current match and could be accommodated by ℓ alongside $\mu(\ell)$, but a "larger" family with a higher priority which also prefers ℓ to its current match cannot be accommodated at ℓ. If the "smaller" family were matched to ℓ, it would be envied by the "large" family (even though the "large" family could not be accommodated at ℓ).

The existence of an envy-free matching is immediate—an empty matching in which all families are matched to the null is envy-free. Consequently, it is straightforward to show the existence of an envy-free matching that is not Pareto-dominated by another envy-free matching. It is, however, not obvious whether an optimal—from the point of view of families—envy-free matching nevertheless exists. Formally, an envy-free matching μ is *family-optimal* if $\mu \succeq \mu'$ for every envy-free matching μ', i.e, μ weakly Pareto dominates every other envy-free matching.

Proposition 5. *There exists a unique family-optimal envy-free matching.*

The existence of the family-optimal envy-free matching provides one natural solution to balancing refugees' preferences and localities' priorities.[20] As we argued above; however, the family-optimal envy-free matching may lead to waste.

We now show that envy-freeness is too restrictive and unnecessarily wasteful in our setting and that certain kinds of innocuous priority violations should be allowed in order to improve refugee family welfare. We illustrate this point with an example.

Example 1 (Inadequacy of envy-freeness in matching with multidimensional constraints)**.** There are three families f_1, f_2, and f_3 and one locality ℓ_1. The priority list of ℓ_1 is $f_1 \rhd_{\ell_1} f_2 \rhd_{\ell_1} f_3$. There are two services and the needs and capacities are displayed below:

$$\nu = \begin{array}{c} \\ s_1 \\ s_2 \end{array} \begin{array}{c} \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \qquad \kappa = \begin{array}{c} \\ s_1 \\ s_2 \end{array} \begin{array}{c} \ell_1 \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{array}.$$

The family-optimal envy-free matching assigns f_1 to ℓ_1 and leaves the other two families unmatched. Envy-freeness precludes assigning f_3 to ℓ_1 because f_2 would then envy f_3. However, f_2 and f_3 are not really in competition since they do not require the same service.

[19]The concept of envy-freeness has been studied in two-sided matching theory (Sotomayor, 1996; Wu and Roth, 2018; Kamada and Kojima, 2018). While Wu and Roth (2018) use the term "envy-free", Sotomayor (1996) and Kamada and Kojima (2018) respectively call the concept "simple" and "fair".

[20]In recent work, Kamada and Kojima (2018) extend Proposition 5 by showing the existence of a family-optimal envy-free matching in a more general setting.

Therefore, a matching in which both f_1 and f_3 are matched to ℓ_1 is more efficient and appears to have no meaningful priority violations. $\square$

5.3 Weakly envy-free matchings

Example 1 shows that even the family-optimal envy-free matching can introduce unnecessary waste by requiring removal of innocuous priority violations. We therefore introduce a solution concept that relaxes envy-freeness in settings with multidimensional constraints by allowing families to be matched to a locality as long as they do not compete with higher-priority families for the services they need.

Recall that a locality ℓ can accommodate a family f alongside set of families $G \subseteq F \setminus \{f\}$ if $\nu_s^f + \sum_{g \in G} \nu_s^g \leq \kappa_s^\ell$ for all $s \in S$. We begin by relaxing this definition.

Definition 8. Locality $\ell \in L$ can *weakly accommodate* family $f \in F$ alongside $G \subseteq F \setminus \{f\}$ if, for all $s \in S$,

$$\text{either} \quad \nu_s^f = 0 \quad \text{or} \quad \nu_s^f + \sum_{g \in G} \nu_s^g \leq \kappa_s^\ell.$$

Definition 8 relaxes the original concept of accommodation by only taking into account the services for which f requires at least one unit. If the needs of families in G exceed the capacity of locality ℓ for some service, f cannot be accommodated alongside G; however, it may still be possible to weakly accommodate f if f does not need any units of that service. An immediate consequence of Definition 8 is that ℓ may be able to weakly accommodate f alongside G even if ℓ cannot satisfy the needs of all families in G. In Example 1, ℓ_1 cannot accommodate f_3 alongside f_1 and f_2 because it cannot even accommodate $\{f_1, f_2\}$. However, ℓ_1 can weakly accommodate f_3 alongside f_1 and f_2 because $\nu_{s_1}^{f_3} = 0$.

We are now in a position to define this section's main solution concept.

Definition 9. Given a matching μ, family $f \in F$ *strongly envies* $f' \neq f$ (with $\ell' = \mu(f')$) if

(i) f prefers f''s locality to its current match, i.e., $\ell' \succ_f \mu(f)$;

(ii) f has a higher priority at ℓ', i.e., $f \rhd_{\ell'} f'$; and

(iii) ℓ' cannot weakly accommodate f' alongside all families with a higher priority than f' at ℓ' that weakly prefer ℓ' to their current matches, i.e., $\{g \in F : g \rhd_{\ell'} f' \text{ and } \ell' \succeq_g \mu(g)\}$.[21]

Definition 10. A matching μ is *weakly envy-free* if no family strongly envies another family.

[21]In fact, if there exist a family f' and a locality ℓ' satisfying condition (iii), then there exists a family $f \in \{g \in F : g \rhd_{\ell'} f' \text{ and } \ell' \succeq_g \mu(g)\}$ for which conditions (i) and (ii) hold. We include conditions (i) and (ii) in Definition 9 in order to emphasize the relationship between envy-freeness and weak envy-freeness.

Weak envy-freeness relaxes envy-freeness by allowing some innocuous priority violations. More precisely, under weak envy-freeness, family f may envy family f' so long as f' does not require any of the services that prevent f from being matched to $\mu(f')$. All envy-free matchings are weakly envy-free but the converse does not hold. In Example 1, the matching that assigns both f_1 and f_3 to ℓ_1 is weakly envy-free, but not envy-free. The reason for this is that even though f_2 has a higher priority than f_3 at ℓ_1, f_3 does not require any units of s_1. Therefore, f_2 envies f_3, but does not strongly envy f_3, when f_3 is matched to ℓ_1.

Let us illustrate how the particular way in which weak envy-freeness relaxes envy-freeness might be relevant in refugee resettlement.

Example 2. There is one locality ℓ (other than the null) that provides 99 units of an abundant service s_1 and one unit of a rare service s_2 (e.g., s_2 could support a rare medical condition): $\kappa^\ell = (99, 1)$. There are two families f_1 and f_2 that each require one unit of the rare service—$\nu^{f_1} = \nu^{f_2} = (0, 1)$—and 100 families $f_3 - f_{102}$ that each require one unit of the abundant service—$\nu^{f_3} = \ldots = \nu^{f_{102}} = (1, 0)$.

Consider the priority profile $f_1 \rhd_\ell f_2, \ldots \rhd_\ell f_{102}$. In the family-optimal envy-free outcome, only family f_1 is matched to ℓ because ℓ cannot accommodate f_2 and f_2 would envy any family f_j matched to ℓ for $j > 2$. This means that all 99 units of s_1 are wasted. In the family-optimal weakly envy-free outcome, in contrast, waste is entirely eliminated as all families except f_2 are matched to ℓ.

Now, consider the priority profile $f_3 \rhd_\ell \ldots \rhd_\ell f_{102} \rhd_\ell f_1 \rhd_\ell f_2$. In the family-optimal envy-free outcome, the only unit of the rare service is wasted because f_{102} would envy f_1 or f_2 if either were matched to ℓ. However, in the family-optimal weakly envy-free outcome, f_1 can be matched to ℓ, which eliminates the waste of the only unit of the rare service. □

Before looking for a family-optimal weakly envy-free matching in the next section, we clarify the logical relationships among the three priority-respecting solution concepts introduced thus far. Envy-freeness and stability are logically independent: Envy-freeness allows waste which stability precludes; however, stability allows for some waste-eliminating priority violations. Weak envy-freeness is logically independent of stability in the same way as envy-freeness is. In contrast, stability only fails due to the presence of waste or via a priority violation so non-wasteful and envy-free matchings are always stable.[22] Perhaps surprisingly, although weak envy-freeness allows for some priority violations, we can establish an analogous relationship among stability, non-wastefulness, and weak envy-freeness.

Proposition 6. *If a matching is non-wasteful and weakly envy-free, then it is stable.*

[22] Kamada and Kojima (2018) define stability to be the combination of envy-freeness and non-wastefulness, making their definition more restrictive than ours.

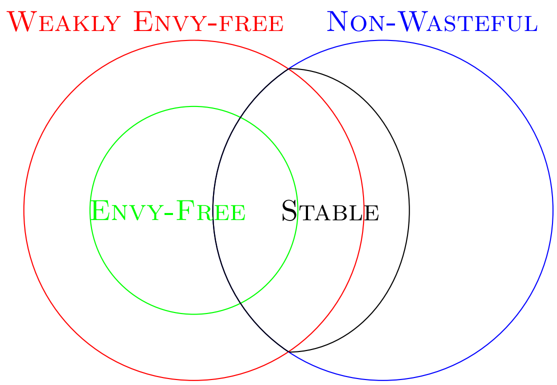

Figure 2: Solution concepts in matching with multidimensional constraints.

Algorithm 3: Cascading Multidimensional Deferred Acceptance (CMDA)

Round $i \geq 1$

Step 1: Every family proposes to its favorite locality that has not permanently rejected it yet.

Step 2: Every locality tentatively accepts a proposing family if the locality can weakly accommodate the family alongside all families with a higher priority that are proposing to or have been permanently rejected by that locality. Otherwise the locality permanently rejects the family.

Step 3: If at least one family has been permanently rejected in Step 2, continue to Round $i+1$. Otherwise permanently match every family to the locality to which the family is proposing and end.

Proposition 6, combined with the possible nonexistence of a stable matching, formally establishes that a weakly envy-free and non-wasteful matching may not exist. Figure 2 summarizes the relationships among the solution concepts.

5.4 A family-optimal weakly envy-free mechanism

We now prove the existence of the (unique) family-optimal weakly envy-free matching by introducing an algorithm that finds this matching in polynomial time (Algorithm 3). In each round of our Cascading Multidimensional Deferred Acceptance (CMDA) algorithm (Algorithm 3), all families propose to their favorite localities that have not permanently rejected them yet. A locality ℓ permanently rejects a family f if ℓ cannot weakly accommodate f alongside families with a higher priority from which ℓ has received a proposal (in this round or a previous one). By construction, families that have already proposed to ℓ can only be matched to ℓ or a less-preferred locality, which implies that if ℓ permanently rejects f then f cannot be matched to ℓ in any weakly envy-free matching. If f is not permanently rejected, then f continues to propose to ℓ in the next round. In each round of the CMDA algorithm, at least one family is permanently rejected so the algorithm eventually terminates. The CMDA algorithm matches each family to its most-preferred locality in any weakly envy-free matching, which yields the following result.

Theorem 5. *The output of the CMDA algorithm is the unique family-optimal weakly envy-free matching.*

Since weak envy-freeness allows for some priority violations, the existence of the family-optimal weakly envy-free matching, which is directly implied by Theorem 5, does not follow from the existence of the family-optimal envy-free matching unless $|S| = 1$ (Wu and Roth, 2018; Kamada and Kojima, 2018).

We now illustrate the CMDA algorithm and discuss its incentive properties.

Example 3. There are four families, four localities, and one service. The preferences, priorities, service needs, and service capacities are

$$\begin{array}{llll} \succ_{f_1}: \ell_2, \ell_1, \dots & \succ_{f_2}: \ell_1, \ell_3, \ell_4, \dots & \succ_{f_3}: \ell_1, \ell_2, \dots & \succ_{f_4}: \ell_1, \ell_3, \dots \\ \rhd_{\ell_1}: f_1, f_2, f_3, f_4 & \rhd_{\ell_2}: f_3, f_1, \dots & \rhd_{\ell_3}: f_4, f_2, \dots & \rhd_{\ell_4}: \dots \end{array}$$

$$\nu = \begin{matrix} & \begin{matrix} f_1 & f_2 & f_3 & f_4 \end{matrix} \\ s_1 & \begin{pmatrix} 1 & 2 & 1 & 1 \end{pmatrix} \end{matrix} \qquad \kappa = \begin{matrix} & \begin{matrix} \ell_1 & \ell_2 & \ell_3 & \ell_4 \end{matrix} \\ s_1 & \begin{pmatrix} 2 & 1 & 2 & 5 \end{pmatrix} \end{matrix}.$$

It can be verified that this market does not have a stable matching. Its family-optimal weakly envy-free matching is

$$\begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ \ell_1 & \ell_4 & \ell_2 & \ell_3 \end{pmatrix}.$$

The family-optimal weakly envy-free matching can be calculated using the CMDA algorithm, which is displayed in Table 1.[23]

In Round 1, all families propose to their favorite localities. Locality ℓ_1 has two units of capacity and receives proposals from families f_2, f_3, and f_4. As family f_2 has the highest priority and needs two units of the service, f_2 is tentatively accepted while the other two families are permanently rejected. In Round 2, families f_3 and f_4 propose to their second-choice localities ℓ_2 and ℓ_3 respectively. Families f_1 and f_3 compete for the single unit of service capacity in locality ℓ_2. Family f_3 is tentatively accepted as f_3 has a higher priority

[23] Since there is only one service, this is also the family-optimal envy-free matching.

Round 1				Round 2				Round 3				Round 4				Round 5			
f_1	→	ℓ_2	✓	f_1	→	ℓ_2	✗	f_1	→	ℓ_1	✓	f_1	→	ℓ_1	✓	f_1	→	ℓ_1	✓
f_2	→	ℓ_1	✓	f_2	→	ℓ_1	✓	f_2	→	ℓ_1	✗	f_2	→	ℓ_3	✗	f_2	→	ℓ_4	✓
f_3	→	ℓ_1	✗	f_3	→	ℓ_2	✓	f_3	→	ℓ_2	✓	f_3	→	ℓ_2	✓	f_3	→	ℓ_2	✓
f_4	→	ℓ_1	✗	f_4	→	ℓ_3	✓	f_4	→	ℓ_3	✓	f_4	→	ℓ_3	✓	f_4	→	ℓ_3	✓

Table 1: CMDA algorithm applied to Example 3.

than f_1; therefore f_1 is permanently rejected. In Round 3, family f_1 proposes to its second-choice locality ℓ_1 and competes with f_2. As locality ℓ_2 has a capacity of two units of the service but families f_1 and f_2 jointly need three units, only the family with the highest priority (i.e., f_1) is tentatively accepted. Hence, locality ℓ_1 permanently rejects family f_2. In Round 4, family f_2 proposes to locality ℓ_3 and is also permanently rejected since f_4 has a higher priority than f_2. In Round 5, f_2 finally proposes to ℓ_4. There are no permanent rejections; therefore, all families are now permanently matched and the algorithm ends. □

Example 3 sheds light on two important aspects of the CMDA algorithm. First, Example 3 shows how some units of capacity can remain unused. In Round 1, families f_3 and f_4 are permanently rejected because f_2 is taking both units of locality ℓ_1's service capacity. In Round 3, however, family f_2 is permanently rejected by locality ℓ_1 because ℓ_1 receives a proposal from f_1. As family f_1 only needs one unit of the service, locality ℓ_1's second unit of service capacity remains unused although this unit could be used by families f_3 or f_4. In the case when there is only one service, the maximum number of units that remain unused in each locality is equal to the highest need of any family minus one. In our example, the highest need is two units (family f_2's need) so at most one unit of capacity may end up unused at each locality. If there are multiple services, this bound remains valid for at least one service in each locality. The number of unused units of other services may be larger.

Second, Example 3 also reveals that the CMDA algorithm is not strategy-proof. To see this, suppose that family f_2 reports locality ℓ_3 to be its first choice. Then, all families are tentatively accepted in the first round since locality ℓ_1 has two units of service capacity and families f_3 and f_4 each require one unit; f_1 and f_2 are the only families that propose to ℓ_2 and to ℓ_3 respectively. The CMDA outcome is

$$\begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ \ell_2 & \ell_3 & \ell_1 & \ell_1 \end{pmatrix}.$$

The following result is an immediate consequence of this observation.

Proposition 7. *There is no strategy-proof and family-optimal weakly envy-free mechanism.*

The driving force behind Proposition 7 is that the acceptance rule (choice function) of lo-

Algorithm 4: THRESHOLD MULTIDIMENSIONAL DEFERRED ACCEPTANCE (TMDA)

Round $i \geq 1$

Step 1: Every family f proposes to its favorite locality ℓ that has not permanently rejected f yet.

Step 2: Every locality ℓ permanently rejects any proposing family f if f's priority rank among all families that are proposing to ℓ is strictly greater than f's threshold at ℓ (calculated by Algorithm 5).

Step 3: If at least one family has been permanently rejected in Step 2, continue to Round $i + 1$. Otherwise permanently match every family to the locality to which the family is proposing and end.

calities induced by the CMDA algorithm does not satisfy the cardinal monotonicity condition (Alkan, 2002; Alkan and Gale, 2003; Fleiner, 2003; Hatfield and Milgrom, 2005).[24] Cardinal monotonicity requires that the number of tentatively accepted families grows monotonically with the number of proposing families. Cardinal monotonicity is crucial for designing strategy-proof matching mechanisms (Hatfield and Milgrom, 2005). Let us see how cardinal monotonicity is violated in Example 3. If families f_3 and f_4 were to propose to locality ℓ_1, they would be tentatively accepted. However, if families f_3, f_4, and f_2 were to propose to locality ℓ_1 together, then only family f_2 would be tentatively accepted. Subsequent to family f_3's permanent rejection, there is a rejection chain that leads to family f_2's being permanently rejected by locality ℓ_1. At the same time, having been permanently rejected by locality ℓ_1, family f_4 ends up competing with f_2 at ℓ_3. Therefore, f_2 is not only permanently rejected by its first-choice locality, but also faces more competition in its second-choice locality. As a result, f_2 has the incentive to misreport its preferences and propose directly to ℓ_3 in order avoid being adversely affected by the two rejection chains. If cardinal monotonicity is satisfied, then f_2's proposal can create at most one rejection chain.

5.5 A weakly envy-free and strategy-proof mechanism

In this section, we introduce the Threshold Multidimensional Deferred Acceptance (TMDA) algorithm (Algorithm 4). We show that the TMDA mechanism is weakly envy-free and strategy-proof. A direct consequence of Proposition 7 is that the TMDA algorithm will not always produce the family-optimal weakly envy-free matching, but, in Online Appendix A.1

[24] In other contexts, cardinal monotonicity has been referred to as "size monotonicity" (Alkan and Gale, 2003) and the "Law of Aggregate Demand" (Hatfield and Milgrom, 2005).

Algorithm 5: Threshold Calculator

For every locality $\ell \in L$, let Π_ℓ be the set of families that are currently proposing to ℓ.

Step 1: For every family $f \in F$ and every locality $\ell \in L$, calculate the *temporary threshold* of f at ℓ, denoted by $\widetilde{\theta}_\ell^f$, as follows:

- If ℓ can weakly accommodate f alongside $\widehat{F}_\ell^f$, let $\widetilde{\theta}_\ell^f = \infty$.
- If ℓ cannot weakly accommodate f alongside $\Pi_\ell \cap \widehat{F}_\ell^f$, let $\widetilde{\theta}_\ell^f = 0$.
- Otherwise, find the unique $n \in \mathbb{Z}_{>0}$, such that
  (i) ℓ can weakly accommodate f alongside all sets of families $G \subseteq \widehat{F}_\ell^f$ such that $|G| = n-1$ and $(\Pi_\ell \cap \widehat{F}_\ell^f) \subseteq G$, and
  (ii) ℓ cannot weakly accommodate f alongside a set of families $G' \subseteq \widehat{F}_\ell^f$ such that $|G'| = n$ and $(\Pi_\ell \cap \widehat{F}_\ell^f) \subseteq G'$;

  and let $\widetilde{\theta}_\ell^f = n$.

Step 2: For every family $f \in F$ and every locality $\ell \in L$, calculate the *threshold* of f at ℓ, θ_ℓ^f, as follows:

- If $\widetilde{\theta}_\ell^f = \infty$, let $\theta_\ell^f = \infty$.
- Otherwise, let $\theta_\ell^f = \min_{g \in \widehat{F}_\ell^f \cup \{f\}} \widetilde{\theta}_\ell^g$.

we show that both the CMDA and TMDA algorithms are guaranteed to match a minimum number of families (Proposition 8).

The TMDA algorithm follows the structure of the CMDA algorithm. In each round, every family proposes to its most preferred locality from which the family has not yet been permanently rejected. Localities tentatively accept some proposals and permanently reject others. This process continues until all proposals are tentatively accepted. In that round, all families are permanently matched to the last locality to which they proposed and the algorithm ends.

The key part of the TMDA algorithm is the choice rule that decides whether a proposal is tentatively accepted or permanently rejected (Algorithm 5). In order to ensure that the TMDA algorithm is strategy-proof, the choice rule of localities needs to satisfy cardinal monotonicity. That is, for each locality, the choice rule ensures that as the set of proposing families expands, the number of families that are tentatively accepted by the locality weakly increases. The choice rule could therefore result in fewer families being tentatively than the locality can weakly accommodate. Consequently, the matching produced by the TMDA algorithm may not be family-optimal in general and additional capacity may remain unsused as a result.

At a high level, the choice rule works as follows (see Step 2 of Algorithm 4). In every round, every locality assigns every family (not just those that are proposing) a *threshold*. For every proposing family, the locality then compares the family's threshold with its priority rank among proposing families. The family is tentatively accepted so long as its priority rank does not exceed its threshold, otherwise the family is permanently rejected.

The details of the threshold calculation are more involved (Algorithm 5). First, every locality ℓ assigns every family f a *temporary* threshold (Step 1 of Algorithm 5). If locality ℓ can weakly accommodate f alongside all higher-priority families $\widehat{F}_\ell^f$ (whether or not they are proposing), f's temporary threshold at ℓ is ∞ (ensuring that ℓ will tentatively accept a proposal from f no matter which other families are proposing). If ℓ cannot weakly accommodate f alongside all higher-priority families $\widehat{F}_\ell^f$ that are currently proposing to ℓ, f's temporary threshold at ℓ is 0 (ensuring that ℓ will permanently reject any future proposal from f since matching f to ℓ would create strong envy from at least one of the families that are currently proposing). Otherwise, f's threshold is a positive integer, calculated as follows. We consider all sets of higher-priority families that contain those higher-priority families that are currently proposing to ℓ. Then, we assign f a temporary threshold n for which (i) ℓ can weakly accommodate f alongside all such sets containing $n-1$ families but (ii) ℓ cannot weakly accommodate f alongside at least one such set of n families. The threshold of family f at ℓ is the smallest temporary threshold among its own and the temporary thresholds

		Round 1			Round 2			Round 3			Round 4			Round 5		
f_1	$\rightarrow$	ℓ_2	[1]	✓	ℓ_2	[0]	✗	ℓ_1	$[\infty]$	✓	ℓ_1	$[\infty]$	✓	ℓ_1	$[\infty]$	✓
f_2	$\rightarrow$	ℓ_1	[1]	✓	ℓ_1	[1]	✓	ℓ_1	[0]	✗	ℓ_3	[0]	✗	ℓ_4	$[\infty]$	✓
f_3	$\rightarrow$	ℓ_1	[0]	✗	ℓ_2	$[\infty]$	✓	ℓ_2	$[\infty]$	✓	ℓ_2	$[\infty]$	✓	ℓ_2	$[\infty]$	✓
f_4	$\rightarrow$	ℓ_1	[0]	✗	ℓ_3	$[\infty]$	✓	ℓ_3	$[\infty]$	✓	ℓ_3	$[\infty]$	✓	ℓ_3	$[\infty]$	✓

Table 2: TMDA algorithm applied to the market from Example 3 with true reporting. Thresholds are in square brackets.

of all families with a higher priority than f at ℓ.[25] The calculation of thresholds can be done polynomial time, which ensures that the TMDA algorithm is practical even for large markets.[26]

Theorem 6. *The TMDA mechanism is strategy-proof and weakly envy-free.*

In Example 3, the TMDA algorithm follows the same deferred acceptance procedure and produces the same matching as the CMDA algorithm. The TMDA algorithm for Example 3 is displayed in Table 2. In Round 1, families f_2, f_3, and f_4 propose to locality ℓ_1. The thresholds of families f_3 and f_4 are 0 since locality ℓ_1 cannot weakly accommodate either one of those families alongside f_2. Hence, families f_3 and f_4 are permanently rejected by locality ℓ_1. In Rounds 2-4, families f_1 and f_2 are assigned a threshold of 0 (at locality ℓ_2 and at localities ℓ_1 and ℓ_3 respectively) and are permanently rejected by these localities. The algorithm ends in Round 5, in which no permanent rejections occur because every family's threshold is infinity.[27]

Even though both algorithms produce the same matching, the TMDA algorithm removes family f_2's incentive to misreport its preferences that f_2 has in the CMDA algorithm. Suppose that family f_2 manipulates its preferences report to $\succ_{f_2}$: $\ell_3, \ell_4, \ldots$. Under the CMDA algorithm, such a manipulation allowed family f_2 to be matched to locality ℓ_3 instead of ℓ_4. This is no longer possible in the TMDA algorithm as we illustrate in Table 3.

In Round 1, following family f_2's manipulation, families f_3 and f_4 propose to ℓ_1. Locality ℓ_1 can weakly accommodate both families but cannot weakly accommodate either one alongside f_2; therefore the threshold of both families is 1. Locality ℓ_1 therefore tentatively accepts

[25]This step precludes any priority violation among families with a finite threshold, which is essential to preserve cardinal monotonicity; however, the matching produced by the TMDA algorithm may still not be envy-free as families with an infinite threshold can still be envied by higher-priority families with finite thresholds.

[26]Take each service for which f needs at least one unit one at a time and order the families with a higher priority from largest to smallest need for that service. Starting from the family with the largest need, add one family at a time until the total need for that service (including f's) exceeds the capacity. A number of families is obtained in this way for each service, n is the minimum among these numbers.

[27]Example 3 is simple and aimed at illustrating the main difference between CMDA and TMDA. For a full worked out example of the TMDA algorithm, see Online Appendix B.3.

Round 1					Round 2					Round 3				
f_1	→	ℓ_2	[1]	✓	f_1	→	ℓ_2	[1]	✓	f_1	→	ℓ_2	[1]	✓
f_2	→	ℓ_3	[1]	✓	f_2	→	ℓ_3	[0]	✗	f_2	→	ℓ_4	[∞]	✓
f_3	→	ℓ_1	[1]	✓	f_3	→	ℓ_1	[1]	✓	f_3	→	ℓ_1	[1]	✓
f_4	→	ℓ_1	[1]	✗	f_4	→	ℓ_3	[∞]	✓	f_4	→	ℓ_3	[∞]	✓

Table 3: TMDA algorithm applied to the market from Example 3 with a misreport. Thresholds are in square brackets.

the proposing family with the highest priority (f_3) but permanently rejects the one with the second-highest priority (f_4). This is the key step in which TMDA differs from CMDA: locality ℓ_1 permanently rejects family f_4 in Round 1 whether or not it receives a proposal from f_2. In Round 2, families f_2 and f_4 propose to locality ℓ_3. As family f_4 has the highest priority for ℓ_3, its threshold is ∞ and f_4 is tentatively accepted by ℓ_3. In contrast, locality ℓ_3 cannot weakly accommodate family f_2 alongside f_4; therefore family f_2's threshold is 0 and f_2 is permanently rejected by ℓ_3. In Round 3, family f_2 proposes to locality ℓ_4 and the algorithm ends as no family is permanently rejected. Note that family f_2's manipulation has benefited f_1 and f_3, which are matched to their first- rather than second-choice localities. However, family f_2 has not managed to benefit from its own manipulation and has remained matched to locality ℓ_4.

One might be concerned that Theorem 6 does not provide a lower bound on the efficiency of the TMDA mechanism. In Online Appendix A.1, we derive a lower bound for the efficiency of the TMDA algorithm (Proposition 8). Finally, we provide a way to improve the efficiency of the TMDA mechanism without affecting its properties (Algorithm 7). Before the TMDA algorithm is run, we identify family-locality pairs that will necessarily be matched together. For every such pair, the family *clinches* the locality and is no longer considered by less-preferred localities. In the TMDA algorithm, clinching may allow us to raise the thresholds of the remaining families. As a result, the matching found by the TMDA with Clinching (TMDAC) mechanism weakly Pareto dominates the matching found by the TMDA mechanism (see Online Appendix A.2 for details).

6 Practical implementation

We now discuss how the mechanisms described in this paper can be put into practice. We first remark on the infrastructure required to elicit refugees' preferences and localities' priorities; then, we give a qualitative comparison of our mechanisms and explain how a resettlement agency might choose among them.

6.1 Collecting preferences and priorities

Eliciting preferences of refugees over localities at first appears to be a daunting task. Indeed, in public school choice, the number of options is relatively small, and yet parents still face difficulties in forming and expressing their preferences (Hastings et al., 2007; Corcoran et al., 2018). In resettlement matching, refugees may in principle need to compare hundreds of localities (e.g., roughly 250 in the U.S.); this would make constructing complete rank-order lists infeasible. Moreover, refugees may not be able to determine precise preferences over individual localities because they lack necessary information, such as an understanding of localities' labor markets and opportunities. Instead, refugees would be asked to rank (or value) the *properties* of localities that are important to them: for example, proximity to a city, low crime, presence of a co-ethnic or a co-religious community (Jones and Teytelboym, 2017b). The resettlement agency can then combine refugees' expressed preferences over characteristics with data on localities to infer a likely preference profile over localities (see, for example, Wiswall and Zafar (2017), who use the hypothetical choice methodology to estimate preferences for workplace attributes).

Eliciting localities' priorities is more straightforward. Resettlement agencies might simply ask localities to list and weight the characteristics of families that they would prefer to resettle. Agencies might want to exclude some characteristics from the ranking process (e.g., race or religion), and expect localities to resettle any family that does not violate its capacity or integration service constraints. Alternatively, localities might want to rank families according to features that are difficult for localities themselves to estimate (e.g., likelihood of employment, see Bansak et al. (2018); Trapp et al. (2018)); in that case, the agency could estimate these characteristics on behalf of the localities and assign priority rankings accordingly.

6.2 Choosing the right mechanism

We have described four mechanisms for refugee resettlement that satisfy different design criteria. As in school choice and other matching market design settings, the choice of which mechanism to use depends on the institutional context (see Section 2), as well as policy goals regarding the impact of refugees' preferences and localities' priorities. Additionally, the choice of mechanism may depend on the structure of preferences and priorities in the market.

If localities' priorities are not of first-order relevance for the resettlement agency, then the agency should consider using MTTC or MTTCE—both of which focus entirely on refugee preferences. In that case, the main decision for the resettlement agency reduces to whether to take an endowment into account. The endowment matters because it guarantees that every family will be placed in a locality that it weakly prefers to its endowment. If an agency does not use an endowment, then the allocation under the MTTC mechanism is entirely driven by refugee preferences. However, an agency may be reluctant to forgo specifying an endowment for at least two reasons. First, the agency may pursue objectives other than simply satisfying refugees' preferences, e.g., maximizing refugees' employment. In this case, the endowment could reflect that objective. Second, the agency may be concerned that preferences are elicited imperfectly and the endowment could insure against particularly poor matches.

In most resettlement contexts, however, localities' priorities are also important. Weak envy-freeness ensures that localities are matched to the highest-priority families that also rank them highly. Hence, weak envy-freeness captures respect for the priorities of localities while taking their multidimensional constraints into account. In choosing between our two weakly envy-free mechanisms, CMDA and TMDA, the resettlement agency must trade-off the concerns about preference manipulation (which TMDA mitigates) with the desire for efficiency gains (under CMDA).

If either (i) refugees' preferences are heterogeneous, but locality priorities are (fairly) homogeneous or (ii) refugees' preferences are (fairly) homogeneous, but locality priorities are heterogeneous, both CMDA and TMDA mechanisms are close to a serial dictatorship. However, the threshold calculator in the TMDA mechanism can be sensitive to small heterogeneity of priorities or preferences thereby potentially causing unnecessary efficiency losses in order to guarantee strategy-proofness. Since manipulations are likely going to be difficult to find in these cases, it might be preferable to use the family-optimal CMDA mechanism instead.

However, the TMDA mechanism might be appropriate if both refugees' preferences and localities' priorities are highly heterogeneous, but correlated with each other, as a weakly envy-free matching produced in this case would be assortative. Note that under preference-priority correlation, the threshold calculator in the TMDA algorithm is unlikely to assign low thresholds at localities that families rank highly (because those families are then also highly-ranked). Then, the TMDA mechanism is unlikely to generate substantial inefficiency.

Finally, if localities' priorities and refugees' preferences are highly both heterogeneous and uncorrelated, then CMDA mechanism might be highly manipulable and the TMDA mechanism might be highly inefficient. In this case, we suggest a two-step process. First, the agency could calculate an initial endowment based on priorities, which could themselves be stated by localities, based on observable characteristics, or a combination of both. Second, the agency could incorporate refugees' preferences by using the MTTCE mechanism. The heterogeneity in refugees' preferences creates opportunities for Pareto-improving chains that the MTTCE algorithm could find. The two-step process also respects priorities as families are endowed with localities for which they have a high priority. Therefore, a family with a high priority for a locality would only move if it is willing to give up its priority for a place at another locality.[28]

7 Conclusion

Refugee resettlement presents a real opportunity for marketplace design: policymakers and resettlement agencies are already working with market design experts to improve matching outcomes in ways that—if we do this work well—stand to improve the lives of millions of disenfranchised people worldwide (Andersson, 2017; Jones and Teytelboym, 2017b; Kominers et al., 2017; Roth, 2018). Recent efforts have focused on maximizing short-run employment outcomes; we show how to take this work further by integrating refugees' preferences and localities' priorities into the assignment process. As we have highlighted, the trade-off among respecting priorities of localities, maximizing refugee welfare, and strategy-proofness is exacerbated by the multidimensional constraint structure present in refugee resettlement matching.

We proposed four matching mechanisms for refugee resettlement: MTTC, which modifies the classical Top Trading Cycles (TTC) algorithm to account for multidimensional resource constraints—and, like TTC, is strategy-proof and finds a Pareto efficient outcome; MTTCE, which further extends MTTC to incorporate endowments; CMDA, a variant of deferred acceptance that is weakly envy-free and family-optimal in our context; and TMDA, which trades off some efficiency relative to CMDA in exchange for strategy-proofness. We have also provided some intuition for how policymakers would choose among these mechanisms in practice, and how they would elicit the preference and priority information needed to use them.

Incorporating refugees' preferences into refugee matching will serve not just to improve the quality of assignment outcomes, but also to give refugees more agency in the resettlement process. Moreover, collecting information about refugees' preferences will enable us to better understand what constitutes a high-quality refugee–locality match. Locality priorities are similarly important: if we want localities to be willing to host a large number of refugees, then we must do what we can to respond to their desires and constraints. The hope is that a well-designed resettlement matching system will increase localities' overall willingness to host refugees, boosting resettlement overall.

[28] This idea is loosely akin to that of the "stable improvement cycles mechanism" proposed for school choice (Erdil and Ergin, 2008).

As we implement mechanisms like those described in this paper, it will be essential to test empirically how much the theoretical trade-offs we characterize manifest in actual preference data. Additionally, we will need to think carefully about how static matching frameworks should account for the dynamic nature of refugee arrival and resettlement (Andersson et al., 2018; Caspari, 2019).

Refugee resettlement matching provides an opportunity for market design to make a real difference into people's lives. While the long-term success of resettlement programs depends on many factors, improving match quality will make sure that as many refugees as possible get off to a good start.

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Appendix: Proofs

Proof of Proposition 1

Recall that μ^i is the current matching in Round i for the MTTC algorithm. We first verify that the output of the MTTC mechanism does not violate any capacity constraint; that is, for all $\ell \in L$, ℓ can accommodate $\mu^{MTTC}(\ell)$. Notice that, in any Round $i \geq 1$, a locality ℓ permanently rejects any family that ℓ cannot accommodate alongside $\mu^i(\ell)$. Therefore, if ℓ can accommodate $\mu^i(\ell)$, then ℓ can accommodate $\mu^{i+1}(\ell)$. Since ℓ can accommodate $\mu^1(\ell) = \varnothing$, it follows by induction that ℓ can accommodate the families currently matched to ℓ in every Round $i \geq 1$. As μ^{MTTC} is the current matching in the last round, ℓ can accommodate $\mu^{MTTC}(\ell)$. We next show that the MTTC mechanism is Pareto-efficient (PE) and strategy-proof (SP).

Proof of (PE) For every Round $i = 1, \ldots, N$, let F^i be the set of families that have been permanently matched by the end of Round $i-1$. (Note that $F^1 = \varnothing$ and $F^{N+1} = F$.) The proof proceeds by induction, with the following hypothesis: there does not exist any matching μ such that $\mu(f) \succeq_f \mu^i(f)$ for all $f \in F^i$ and $\mu(f) \succ_f \mu^i(f)$ for some $f \in F^i$. Our induction hypothesis trivially holds for $i = 1$ since $F^1 = \varnothing$. We now show that if our induction hypothesis holds for some $i = 1, \ldots, N$, then it also holds for $i+1$.

Towards a contradiction, suppose there exists a matching μ such that $\mu(f) \succeq_f \mu^{i+1}(f)$ for all $f \in F^{i+1}$ and $\mu(f) \succ_f \mu^{i+1}(f)$ for some $f \in F^{i+1}$. Note that, for all $f \in F^i$, $\mu^i(f) = \mu^{i+1}(f)$. If, for some $f \in F^i$, $\mu(f) \succ_f \mu^{i+1}(f)$, the induction hypothesis implies that there exists of a family $f' \in F^i$ such that $\mu^{i+1}(f') \succ_{f'} \mu(f')$, a contradiction. Therefore, $\mu^{i+1}(f) = \mu(f)$ for all $f \in F^i$. Note that, for all $f \in F^{i+1} \setminus F^i$, f is permanently matched to $\mu^{i+1}(f)$ in Round i; therefore $\mu^{i+1}(f)$ is the locality f prefers among those that have not permanently rejected f. If, for some $f \in F^{i+1} \setminus F^i$, $\mu(f) \succ_f \mu^{i+1}(f)$, then $\mu(f)$ has permanently rejected f so it must be that $\mu(f)$ cannot accommodate f alongside $\mu^i(\mu(f))$. Therefore, there exists a family $f' \in F^i$ such that $\mu^i(f') \neq \mu(f')$, a contradiction.

By induction, there does not exist any matching μ such that $\mu(f) \succeq_f \mu^{N+1}(f)$ for all $f \in F^{N+1}$ and $\mu(f) \succ_f \mu^{N+1}(f)$ for some $f \in F^{N+1}$. As $\mu^{N+1} = \mu^{MTTC}$ and $F^{N+1} = F$, this implies that μ^{MTTC} is Pareto-efficient.

Proof of (SP) The result is implied by the fact that the MTTCE mechanism is strategy-proof (Theorem 3) and the fact that $\mu^{MTTCE} = \mu^{MTTC}$ when every family is endowed $\emptyset$ (Proposition 3). □

Proof of Theorem 1

The proof is by counterexample. There are four families, four localities, and one service. The endowment is

$$\mu^E = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ \ell_2 & \ell_2 & \ell_3 & \ell_4 \end{pmatrix}.$$

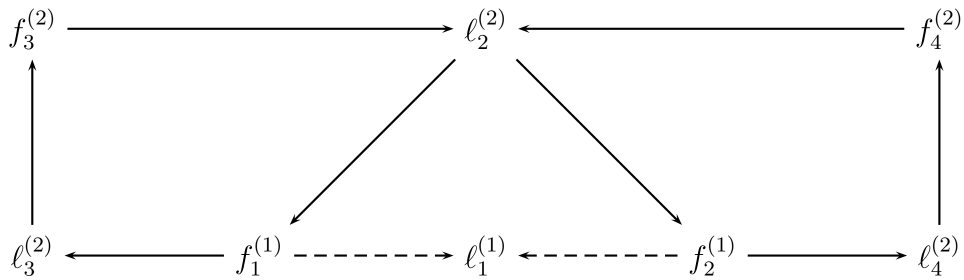

Figure 3: Counterexample for the proof of Theorem 1. $f \to \ell$: ℓ is f's first choice. $f \dashrightarrow \ell$: ℓ is f's second choice and f prefers ℓ to its endowment. $\ell \to f$: ℓ is f's endowment. Superscripts denote needs and capacities, respectively.

The preferences of families are as follows:

$$\succ_{f_1}: \ell_3, \ell_1, \boldsymbol{\ell_2}, \ldots \quad \succ_{f_2}: \ell_4, \ell_1, \boldsymbol{\ell_2}, \ldots \quad \succ_{f_3}: \ell_2, \boldsymbol{\ell_3}, \ldots \quad \succ_{f_4}: \ell_2, \boldsymbol{\ell_4}, \ldots,$$

where a family's endowment locality is denoted in boldface. The service needs and service capacities are

$$\nu = \begin{array}{c} \\ s_1 \end{array} \begin{array}{c} \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \left(\begin{array}{cccc} 1 & 1 & 2 & 2 \end{array} \right) \end{array} \qquad \kappa = \begin{array}{c} \\ s_1 \end{array} \begin{array}{c} \begin{array}{cccc} \ell_1 & \ell_2 & \ell_3 & \ell_4 \end{array} \\ \left(\begin{array}{cccc} 1 & 2 & 2 & 2 \end{array} \right) \end{array}.$$

For ease of exposition, we illustrate this counterexample in Figure 3.

We show that there are two individually rational (IR) and chain-efficient (CE) matchings:

$$\mu = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ \ell_1 & \ell_4 & \ell_3 & \ell_2 \end{pmatrix} \quad \text{and} \quad \mu' = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ \ell_3 & \ell_1 & \ell_2 & \ell_4 \end{pmatrix}.$$

It is easy to check that matchings μ and μ' are IR and Pareto-efficient; hence they are IR and CE. It remains to show that no other IR matching is CE.

First, consider the case where f_1 is matched to ℓ_3. Then, f_3 must be matched to ℓ_2 and f_4 must be matched to ℓ_4. Therefore, ℓ_1 must be matched to f_2, yielding μ'; hence μ' is the only IR (and CE) matching where f_1 is matched to ℓ_3. Analogous reasoning allows us to conclude that μ is the only IR (and CE) matching where f_2 is matched to ℓ_4.

Second, consider the case where f_1 is matched to its endowment locality ℓ_2. Then, f_3 and f_4 must also be matched to their respective endowments, ℓ_3 and ℓ_4. Hence, f_2 must be matched to either ℓ_2 or ℓ_1. Matching f_2 to ℓ_2 yields μ^E. Since f_2 prefers ℓ_1 to ℓ_2 and ℓ_1 is not matched to any family, μ^E is wasteful, hence not CE. Matching f_2 to ℓ_1 yields a matching that is not CE as it contains the Pareto-improving chain $f_1 \to \ell_3 \to f_3 \to \ell_2$. Therefore, there is no IR and CE matching where f_1 is matched to ℓ_2. Analogous reasoning allows us to reach the same conclusion for f_2.

Third, consider the case where f_1 is matched to ℓ_1. Then, f_2 must be matched to either ℓ_4 or ℓ_2, which by our previous argument yields either μ or μ^E. We therefore conclude that μ and μ' are the only two IR and CE matchings.

Suppose now that f_1 misreports its preferences by ranking ℓ_1 below ℓ_2, i.e., $\succ'_{f_1}: \ell_3, \boldsymbol{\ell_2}, \ldots$. Using analogous reasoning as for the true preference profile, it is easy to check that μ' is the

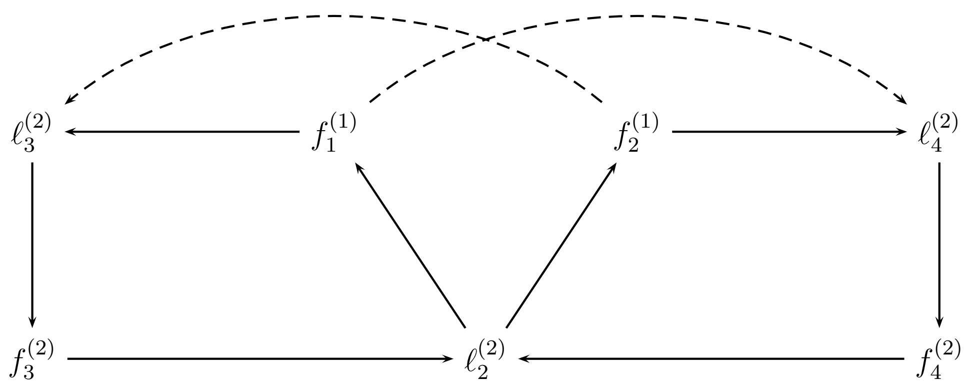

Figure 4: Counterexample for the proof of Proposition 2. $f \rightarrow \ell$: ℓ is f's first choice. $f \dashrightarrow \ell$: ℓ is f's second choice and f prefers ℓ to its endowment. $\ell \rightarrow f$: ℓ is f's endowment. Superscripts denote needs and capacities, respectively.

only IR and CE matching for the manipulated preference profile $(\succ'_{f_1}, \succ_{-f_1})$. Similarly, μ is the only IR and CE matching for the manipulated preference profile $(\succ'_{f_2}, \succ_{-f_2})$ where f_2 misreports its preferences by ranking ℓ_1 below ℓ_2, i.e., $\succ'_{f_2}: \ell_4, \boldsymbol{\ell_2}, \ldots$.

We can now show that no IR and CE mechanism is strategy-proof. Let φ be an IR and CE mechanism. If all families report their preferences truthfully, then either $\varphi(\succ) = \mu$ or $\varphi(\succ) = \mu'$ because μ and μ' are the only two IR and CE matchings. If f_1 reports $\succ'_{f_1}$, then $\varphi(\succ'_{f_1}, \succ_{-f_1}) = \mu'$. Similarly, if f_2 reports $\succ'_{f_2}$, then $\varphi(\succ'_{f_2}, \succ_{-f_2}) = \mu$. If $\varphi(\succ) = \mu$, then $\varphi(\succ'_{f_1}, \succ_{-f_1})(f_1) \succ_{f_1} \varphi(\succ)(f_1)$ but if $\varphi(\succ) = \mu'$, then $\varphi(\succ'_{f_2}, \succ_{-f_2})(f_2) \succ_{f_2} \varphi(\succ)(f_2)$. Therefore φ is not strategy-proof. $\square$

Proof of Proposition 2

The proof is by counterexample. There are four families, three localities, and one service. The endowment is

$$\mu^E = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ \ell_2 & \ell_2 & \ell_3 & \ell_4 \end{pmatrix}.$$

The preferences of families are as follows:

$$\succ_{f_1}: \ell_3, \ell_4, \boldsymbol{\ell_2}, \ldots \quad \succ_{f_2}: \ell_4, \ell_3, \boldsymbol{\ell_2}, \ldots \quad \succ_{f_3}: \ell_2, \boldsymbol{\ell_3}, \ldots \quad \succ_{f_4}: \ell_2, \boldsymbol{\ell_4}, \ldots$$

where a family's endowment locality is denoted in boldface. The service needs and service capacities are

$$\nu = \begin{matrix} & f_1 & f_2 & f_3 & f_4 \\ s_1 & (1 & 1 & 2 & 2) \end{matrix} \qquad \kappa = \begin{matrix} & \ell_2 & \ell_3 & \ell_4 \\ s_1 & (2 & 2 & 2) \end{matrix}.$$

For ease of exposition, we illustrate this counterexample in Figure 4.

We show that there are exactly two matchings that Pareto dominate μ^E:

$$\mu = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ \ell_3 & \ell_3 & \ell_2 & \ell_4 \end{pmatrix} \quad \text{and} \quad \mu' = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ \ell_4 & \ell_4 & \ell_3 & \ell_2 \end{pmatrix}.$$

Matching μ (resp. μ') Pareto dominates μ^E since it matches family f_4 (resp. f_3) to its endowment and make the other three families better-off. We now show that no other matching Pareto dominates μ^E. Towards a contradiction, suppose the existence of a matching $\tilde{\mu}$ that Pareto dominates μ^E and is different from both μ and μ'. Consider the case where $\tilde{\mu}(f_3) \succ_{f_3} \mu^E(f_3)$. Then we must have that $\tilde{\mu}(f_3) = \ell_2$. Therefore, locality ℓ_2 cannot accommodate any other family so $\tilde{\mu}(f_4) = \ell_4$. In turn, locality ℓ_4 cannot accommodate any other family so $\tilde{\mu}(f_1) = \tilde{\mu}(f_2) = \ell_3$ and $\tilde{\mu} = \mu$. Analogously, we have that $\tilde{\mu}(f_4) \succ_{f_4} \mu^E(f_4)$ implies $\tilde{\mu} = \mu'$. Therefore, $\tilde{\mu}(f_3) = \ell_3$ and $\tilde{\mu}(f_4) = \ell_4$. Hence, neither ℓ_3 nor ℓ_4 can accommodate another family so $\tilde{\mu}(f_1) = \tilde{\mu}(f_2) = \ell_2$ and $\tilde{\mu} = \mu^E$, a contradiction.

We have established that μ and μ' are the only two matchings that Pareto dominate μ^E with respect to the true preferences. We now consider consider a preference manipulation by f_1. Suppose that f_1 misreports its true preferences by ranking ℓ_4 below ℓ_2, i.e., f_1 reports $\succ'_{f_1}\colon \ell_3, \boldsymbol{\ell_2}, \ldots$. We claim that if all other families report truthfully, then μ is the only matching that Pareto dominates μ^E with respect to the manipulated preference profile $(\succ'_{f_1}, \succ_{-f_1})$. Suppose again towards a contradiction that there exists a matching $\tilde{\mu}$ that Pareto dominates μ^E and is different from μ. There are three cases. First, if $\tilde{\mu}(f_3) \succ_{f_3} \mu^E(f_3)$, then $\tilde{\mu}(f_3) = \ell_2$ and $\tilde{\mu}(f_4) = \ell_4$. Therefore, we must have that $\tilde{\mu}(f_1) = \tilde{\mu}(f_2) = \ell_3$ so $\tilde{\mu} = \mu$, a contradiction. Second, if $\tilde{\mu}(f_4) \succ_{f_4} \mu^E(f_4)$, then $\tilde{\mu}(f_4) = \ell_2$ and $\tilde{\mu}(f_3) = \ell_3$. Since neither ℓ_2 nor ℓ_3 can accommodate another family, f_1 must be matched to a less-preferred locality than $\mu^E(f_1) = \ell_2$ (according to f_1's manipulated report) hence $\tilde{\mu}$ does not Pareto dominate μ^E, a contradiction. Third, if $\tilde{\mu}(f_3) = \ell_3$ and $\tilde{\mu}(f_4) = \ell_4$, then $\tilde{\mu} = \mu^E$, a contradiction. We therefore conclude that there is no matching that Pareto dominates μ^E for the preference profile $(\succ'_{f_1}, \succ_{-f_1})$ and that is different from μ. By analogous reasoning, one can verify that if f_2 manipulates its preferences by reporting $\succ'_{f_2}\colon \ell_4, \boldsymbol{\ell_2}, \ldots$ and all other families report truthfully, then μ' is the only matching that Pareto dominates μ^E for the preference profile $(\succ'_{f_2}, \succ_{-f_2})$.

We can now show that there is no strategy-proof mechanism that Pareto improves upon μ^E. Let φ be a mechanism that Pareto improves upon μ^E. If all families report their preferences truthfully, then either $\varphi(\succ) = \mu$ or $\varphi(\succ) = \mu'$ because μ and μ' are the only two matchings that Pareto dominate μ^E for the true preference profile. If f_1 reports $\succ'_{f_1}$, then $\varphi(\succ'_{f_1}, \succ_{-f_1}) = \mu$. Similarly, if f_2 reports $\succ'_{f_2}$, then $\varphi(\succ'_{f_2}, \succ_{-f_2}) = \mu'$. If $\varphi(\succ) = \mu$, then $\varphi(\succ'_{f_2}, \succ_{-f_2})(f_2) \succ_{f_2} \varphi(\succ)(f_2)$ but if $\varphi(\succ) = \mu'$, then $\varphi(\succ'_{f_1}, \succ_{-f_1})(f_1) \succ_{f_1} \varphi(\succ)(f_1)$. Therefore, φ is not strategy-proof. □

Proof of Theorem 2

The proof is by counterexample. There are four families, three localities, and two services. The endowment is

$$\mu^E = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ \ell_2 & \ell_2 & \ell_3 & \ell_4 \end{pmatrix}.$$

The preferences of families are as follows:

$$\succ_{f_1}\colon \ell_3, \boldsymbol{\ell_2}, \ldots \qquad \succ_{f_2}\colon \ell_4, \boldsymbol{\ell_2}, \ldots \qquad \succ_{f_3}\colon \ell_2, \ell_4, \boldsymbol{\ell_3}, \ldots \qquad \succ_{f_4}\colon \ell_2, \ell_3, \boldsymbol{\ell_4}, \ldots$$

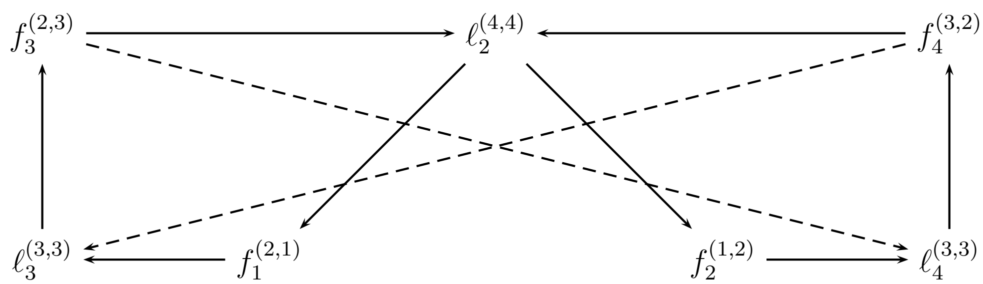

Figure 5: Counterexample for the proof of Theorem 2. $f \to \ell$: ℓ is f's first choice. $f \dashrightarrow \ell$: ℓ is f's second choice and f prefers ℓ to its endowment. $\ell \to f$: ℓ is f's endowment. Superscripts denote needs and capacities, respectively.

where a family's endowment locality is denoted in boldface. The service needs and service capacities are

$$\nu = \begin{array}{c} \\ s_1 \\ s_2 \end{array} \begin{array}{c} \begin{matrix} f_1 & f_2 & f_3 & f_4 \end{matrix} \\ \begin{pmatrix} 2 & 1 & 2 & 3 \\ 1 & 2 & 3 & 2 \end{pmatrix} \end{array} \qquad \kappa = \begin{array}{c} \\ s_1 \\ s_2 \end{array} \begin{array}{c} \begin{matrix} \ell_2 & \ell_3 & \ell_4 \end{matrix} \\ \begin{pmatrix} 3 & 3 & 4 \\ 3 & 3 & 4 \end{pmatrix} \end{array}.$$

For ease of exposition, we illustrate this counterexample in Figure 5.

Observe that μ^E is non-wasteful but not chain-efficient as it contains three Pareto-improving chains:

$$(f_2, \ell_4, f_4, \ell_3, f_3, \ell_2), \quad (f_1, \ell_3, f_3, \ell_4, f_4, \ell_2), \quad \text{and} \quad (f_3, \ell_4, f_4, \ell_3).$$

Executing any one of these Pareto-improving chains on μ^E yields one of the following three matchings:

$$\mu = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ \ell_2 & \ell_4 & \ell_2 & \ell_3 \end{pmatrix}, \quad \mu' = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ \ell_3 & \ell_2 & \ell_4 & \ell_2 \end{pmatrix}, \quad \text{and} \quad \mu'' = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ \ell_2 & \ell_2 & \ell_4 & \ell_3 \end{pmatrix}.$$

We first show that μ, μ', and μ'' are the only matchings that Pareto dominate the endowment μ^E.

Suppose towards a contradiction that there exists a matching $\tilde{\mu}$ that Pareto dominates μ^E. There are three cases.

First, consider the case where $\tilde{\mu}(f_3) = \ell_3$. Then, neither f_1 nor f_4 can be matched to ℓ_3; hence $\tilde{\mu}(f_1) = \ell_2$ and $\tilde{\mu}(f_4) = \ell_4$. Therefore, we must have that $\tilde{\mu}(f_2) = \ell_2$. It follows that $\tilde{\mu} = \mu^E$, a contradiction.

Second, consider the case where $\tilde{\mu}(f_3) = \ell_2$. Then, f_2 cannot be matched to ℓ_2 so $\tilde{\mu}(f_2) = \ell_4$. As a result, f_4 cannot be matched to either ℓ_2 or ℓ_4, hence $\tilde{\mu}(f_4) = \ell_3$. It follows that f_1 cannot be matched to ℓ_3; therefore $\tilde{\mu}(f_3) = \ell_2$ and $\tilde{\mu} = \mu$, a contradiction.

Third, consider the case where $\tilde{\mu}(f_3) = \ell_4$. Neither f_2 nor f_4 can be matched to ℓ_4; therefore $\tilde{\mu}(f_2) = \ell_2$ and we must have that either $\tilde{\mu}(f_4) = \ell_2$ or $\tilde{\mu}(f_4) = \ell_3$. If $\tilde{\mu}(f_4) = \ell_2$, then $\tilde{\mu}(f_1) = \ell_3$ and $\tilde{\mu} = \mu'$. If $\tilde{\mu}(f_4) = \ell_3$, then $\tilde{\mu}(f_1) = \ell_2$ and $\tilde{\mu} = \mu''$, a contradiction.

We have established that μ, μ', and μ'' are the only matchings that Pareto dominate μ^E. Suppose that f_3 misreports its preferences by ranking ℓ_4 below ℓ_3, i.e., by reporting

$\succ'_{f_3}$: $\ell_2, \boldsymbol{\ell_3}, \ldots$, then μ is the unique matching that Pareto dominates μ^E for the manipulated preference profile $(\succ'_{f_3}, \succ_{-f_3})$. Similarly, suppose that f_4 misreports its preferences by ranking ℓ_3 below ℓ_4, i.e., by reporting $\succ'_{f_4}$: $\ell_2, \boldsymbol{\ell_4}, \ldots$, then μ' is the unique matching that Pareto dominates μ^E for the manipulated preference profile $(\succ'_{f_4}, \succ_{-f_4})$.

We can now show that there is no strategy-proof mechanism that Pareto improves upon μ^E. Let φ be a mechanism that Pareto improves upon μ^E. If all families report their preferences truthfully, then $\varphi(\succ) \in \{\mu, \mu', \mu''\}$ because μ, μ', and μ'' are the only three matchings that Pareto dominate μ^E for the true preference profile. If f_3 reports $\succ'_{f_3}$, then $\varphi(\succ'_{f_3}, \succ_{-f_3}) = \mu$. If f_4 reports $\succ'_{f_4}$, then $\varphi(\succ'_{f_4}, \succ_{-f_4}) = \mu'$. If $\varphi(\succ) \in \{\mu, \mu''\}$, then $\varphi(\succ'_{f_4}, \succ_{-f_4})(f_4) \succ_{f_4} \varphi(\succ)(f_4)$. If $\varphi(\succ) \in \{\mu', \mu''\}$, then $\varphi(\succ'_{f_3}, \succ_{-f_3})(f_3) \succ_{f_3} \varphi(\succ)(f_3)$. Therefore, φ is not strategy-proof. □

Proof of Theorem 3

We first verify that the output of the MTTCE mechanism does not violate any capacity constraint; that is, for all $\ell \in L$, ℓ can accommodate $\mu^{MTTCE}(\ell)$. Note that, in any Round $i \geq 1$, the current matching μ^i is only updated when a feasible cycle is carried out. By definition, every family f that moves to locality ℓ as part of a feasible cycle can be accommodated by ℓ alongside all the families that remain at ℓ after the cycle is carried out; therefore, if ℓ can accommodate $\mu^i(\ell)$, then ℓ can accommodate $\mu^{i+1}(\ell)$. By assumption, the endowment does not violate any capacity constraint; hence, by induction, ℓ can accommodate the families currently matched to ℓ in every Round $i \geq 1$. As μ^{MTTCE} is obtained by carrying out feasible cycles from the current matching in the last round of the MTTCE algorithm, then ℓ can accommodate $\mu^{MTTCE}(\ell)$.

We next show that the MTTCE mechanism is individually rational (IR) and strategy-proof (SP).

Proof of (IR) Consider a family f, its endowment ℓ, and let i be the round of the MTTCE algorithm in which f is permanently matched. We need to show that $\mu^{MTTCE}(f) \succeq_f \ell$. By construction, any family that is not permanently matched is currently matched to its endowment; thus f is currently matched to ℓ at the start of Round i, i.e., $f \in \mu^i(\ell)$. Recall that in any round, locality ℓ can accommodate families currently matched to ℓ, in particular ℓ can accommodate $\mu^i(\ell)$. As a result, ℓ has not permanently rejected f. By definition, f points at its most preferred locality that has not permanently rejected f yet. Therefore, f points at a locality that f weakly prefers to ℓ. As f is permanently matched in Round i, f points at $\mu^{MTTCE}(f)$. Therefore, $\mu^{MTTCE}(f) \succeq_f \ell$, which is what IR requires.

Proof of (SP) Consider a family f with true preferences $\succ_f$, let $\succ'_f$ be an alternative report, and fix the reports of all other families to $\succ_{-f}$. Denote by $\ell = \varphi^{MTTCE}(\succ_f, \succ_{-f})(f)$, respectively $\ell' = \varphi^{MTTCE}(\succ'_f, \succ_{-f})(f)$, the locality with which f is matched if it reports $\succ_f$, respectively $\succ'_f$. We need to show that $\ell \succeq_f \ell'$. Let $m = 1, \ldots, N$, respectively $m' = 1, \ldots, N$, be the round in which f gets permanently matched with report $\succ_f$, respectively $\succ'_f$.

Case 1: $m \leq m'$. Whether it reports $\succ_f$ or $\succ'_f$, f is not permanently matched at the start of Round m. Let L_f^m be the set of localities that have not permanently rejected f at the start of Round m (this set is nonempty as we have already shown that a family is never permanently rejected by its endowment). Notice that, in any given round, f's report does not impact whether or not a given locality permanently rejects f; therefore L_f^m is the same whether f reports $\succ_f$ or $\succ'_f$. In addition, permanent rejections are irreversible; therefore, with either report, f will be matched to a locality in L_f^m; therefore, $\ell, \ell' \in L_f^m$. If f reports truthfully, it points at its most preferred locality in L_f^m and is permanently matched to it. Hence, ℓ is f's most preferred locality in L_f^m, which implies that $\ell \succeq_f \ell'$.

Case 2: $m > m'$. As before, observe that, whether it reports $\succ_f$ or $\succ'_f$, f is not permanently matched at the start of Round m' and let $L_f^{m'}$ be the set of localities that have not permanently rejected f at the start of Round m'. If f reports $\succ'_f$, f points at ℓ' in Round m'. A feasible cycle $f \to \ell' \to f_2 \to \ell_2 \to \ldots \to f_n \to \ell_n \to f$ appears and f is permanently matched to ℓ'. By construction, in the MTTCE algorithm, the report of family f affects other families' pointing behavior only after f is permanently matched. Thus, as f is not matched until Round $m > m'$ when it reports $\succ_f$ and every family and locality can be part of at most one cycle, none of $f_2, \ldots, f_n$ and $\ell', \ell_1, \ldots, \ell_n$ are in a cycle in Round m', hence none of them are permanently matched in Round m'.

We next show that none of $f_2, \ldots, f_n$ and $\ell', \ell_1, \ldots, \ell_n$ are permanently matched before the start of Round m. By assumption, the cycle $f \to \ell' \to f_2 \to \ell_2 \to \ldots \to f_n \to \ell_n \to f$ is feasible, which means that for all $j = 2, \ldots, n-1$, ℓ_j can accommodate f_j alongside $\mu^{m'}(\ell_j) \setminus \{f_{j+1}\}$. Therefore, ℓ_j does not permanently reject f_j in any Rejection Stage before Round m. Analogously, ℓ_1 does not permanently reject f and ℓ_n does not permanently reject f_n in any Rejection Stage before Round m. Therefore, in Round m, all families $f_2, \ldots, f_n$ and all localities $\ell', \ell_2, \ldots, \ell_n$ continue to point as they did in Round m', meaning that f is permanently matched to ℓ' if f points at ℓ'. Since ℓ' has not permanently rejected f by Round m, f is permanently matched to either ℓ' or to a more preferred locality. Therefore, f can do no worse by reporting $\succ_f$ than by reporting $\succ'_f$. □

Proof of Proposition 3

We first show that all cycles that appear in the MTTCE algorithm are feasible. Towards a contradiction, suppose that an infeasible cycle $f_1 \to \ell_1 \to f_2 \to \ell_2 \to \ldots \to f_n \to \ell_n \to f_1$ appears in some Round i. Then, there exists $j = 1, \ldots, n$ such that ℓ_j cannot accommodate f_j alongside $\mu^i(\ell_j) \setminus \{f_{j+1}\}$ (letting $f_{n+1} = f_1$). As $\emptyset$ can accommodate all families, $\ell_j \neq \emptyset$ so $\mu^E(\ell_j) = \varnothing$; hence f_{j+1} is not in ℓ_j's endowment. Moreover, f_{j+1} is not permanently matched at the start of Round i as otherwise ℓ_j would not point at f_{j+1}; therefore $f \in \mu^i(\emptyset)$, which implies $f \notin \mu^i(\ell_j)$. Then, $\mu^i(\ell_j) \setminus \{f_{j+1}\} = \mu^i(\ell_j)$; hence ℓ_j cannot accommodate f_j alongside $\mu^i(\ell_j)$. As $\mu^E(\ell_j) = \varnothing$, by construction all families in $\mu^i(\ell_j)$ have been permanently matched to ℓ_j before the start of Round i. Therefore, at the start of Round i, ℓ_j permanently rejects f_j as ℓ_j cannot accommodate f_j alongside all the families permanently matched to ℓ_j. We conclude that f_j does not point to ℓ_j in Round i, a contradiction.

We have showed that all cycles that appear in the MTTCE algorithm are feasible, which implies that every round ends in the Matching Stage. Therefore, the MTTCE algorithm carries out all the cycles that appear in every round and the algorithm never enters the Rejection Stage. As a result, each round of the MTTCE algorithm coincides with the corresponding round of the MTTC algorithm. □

Proof of Theorem 4

Consider an instance in which needs of families are monotonic, priorities of localities are lexicographic, and the endowment is μ^E. Suppose that for this instance the MTTCE mechanism produces the endowment, i.e., $\mu^{MTTCE} = \mu^E$. Since the MTTCE mechanism does not Pareto improve upon this endowment, we need to show that μ^E is chain-efficient. Let N be the total number of rounds of the MTTCE algorithm. The fact that the MTTCE algorithm produces μ^E implies that the current matching is the same in every round: $\mu^E = \mu^1 = \mu^2 = \ldots = \mu^N = \mu^{N+1} = \mu^{MTTCE}$. Let M^* be the set of matchings that can be obtained by starting from μ^E and carrying out exactly one Pareto-improving chain. Now μ^E is chain-efficient if and only if $M^* = \varnothing$; therefore it remains to show that $M^* = \varnothing$.

We proceed by induction with the following hypothesis: if a locality ℓ has permanently rejected a family f by the start of Round i, then for all $\mu^* \in M^*$, $\mu^*(f) \neq \ell$. Our inductive hypothesis trivially holds for $i = 1$ since no permanent rejection occurs before the start of Round 1. Assuming that the inductive hypothesis holds for some $i = 1, \ldots, N$, we show that it holds for $i + 1$.

Consider a family f that has been permanently matched before the start of Round i. Recall that $\mu^E = \mu^i$, so f is permanently matched to its endowment $\mu^E(f)$; therefore f has been permanently rejected by all localities it prefers to $\mu^E(f)$. By the induction hypothesis, we have that, for all $\mu^* \in M^*$, $\mu^*(f) = \mu^E(f)$. We can therefore conclude that all families that are permanently matched before the start of Round i are matched to their endowment in all matchings contained in M^*.

Now consider a locality ℓ that permanently rejects a family f in Round i. We need to show that $\mu^*(f) \neq \ell$ for all $\mu^* \in M^*$. The fact that ℓ permanently rejects f in Round i implies that, at the start of Round i, f is not permanently matched and ℓ has not permanently rejected f yet. There are two cases in which ℓ can permanently reject f: (1) at the beginning of Round i, or (2) in the Rejection Stage of Round i.

Case 1: ℓ permanently rejects f at the beginning of Round i. By definition, ℓ cannot accommodate f alongside all the families permanently matched to ℓ. Since all families that are permanently matched before the start of Round i are matched to their endowment in all matchings contained in M^*, we have that all the families that are permanently matched to ℓ at the start of Round i are also matched to ℓ in all matchings contained in M^*. Therefore, $\mu^*(f) \neq \ell$ for all $\mu^* \in M^*$.

Case 2: ℓ permanently rejects f in the Rejection Stage of Round i. By definition, ℓ cannot accommodate f alongside $\mu^E(\ell) \setminus \{f'\}$ (where f' is the family at which ℓ is pointing, if any). Since a family is never permanently rejected by its endowment, we have that $\ell \neq \mu^E(f)$. By construction of the MTTCE algorithm, a locality does not point at any family if the locality has permanently rejected all families that are not permanently matched. Then the fact that ℓ has not permanently rejected f yet and that f is not permanently matched implies that ℓ *does* point at some family $f' \in F$. By construction, f' has the highest priority among all families that are not permanently matched and have not been permanently rejected by ℓ. We consider two sub-cases: (2.1) $f' \notin \mu^E(\ell)$, and (2.2) $f \in \mu^E(\ell)$.

Sub-case 2.1: $f' \notin \mu^E(\ell)$. As priorities are lexicographic, all families in $\mu^E(\ell)$ have a higher priority than f'. The fact that ℓ points at f' implies that all families in $\mu^E(\ell)$ either have been permanently rejected by ℓ or have been permanently matched to ℓ. As a family

Algorithm 6: Sequential Deferred Acceptance

Construct a directed graph $\mathbb{G}^1$ as follows. Each of the $|F|$ vertices represents a family. For every pair of families (f, f'), let there be a directed edge from f to f' if $\nu^f \neq \nu^{f'}$ and $f \rhd_\ell f'$ for all $\ell \in L$.

For every locality ℓ, set a *counter* $c_\ell^1 = \kappa^\ell$.

Round $i \geq 1$

Let $\widetilde{F}^i$ be the set of families at which no other family is pointing in graph $\mathbb{G}^i$.

Permanently match the families in $\widetilde{F}^i$ to the localities using the family-proposing Deferred Acceptance algorithm setting the capacity of every locality ℓ to its counter c_ℓ^i.

If all families have been permanently matched, end.

Otherwise, construct $\mathbb{G}^{i+1}$ by removing from $\mathbb{G}^i$ all vertices representing families in $\widetilde{F}^i$ and all edges adjacent to them. For every $\ell \in L$, let $\widetilde{F}^i_\ell$ be the set of families that have been permanently matched to ℓ in Round i. Update the counter of every locality ℓ as follows: $c_\ell^{i+1} = c_\ell^i - \sum_{f \in \widetilde{F}^i_\ell} \nu^f$. Continue to Round $i+1$.

cannot be permanently rejected by its endowment, all families in $\mu^E(\ell)$ are permanently matched to ℓ at the start of Round i. Recall that ℓ cannot accommodate f alongside $\mu^i(\ell) \setminus \{f'\}$. Since $\mu^i(\ell) = \mu^E(\ell)$ and $f' \notin \mu^E(\ell)$, ℓ cannot accommodate f alongside $\mu^i(\ell)$. Therefore, ℓ permanently rejects f at the beginning of Round i, contradicting our assumption that the permanent rejection occurs in the Rejection Stage.

Sub-case 2.2: $f' \in \mu^E(\ell)$. Towards a contradiction, suppose there exists a matching $\mu^* \in M^*$ such that $\mu^*(f) = \ell$. Recall that ℓ cannot accommodate f alongside $\mu^E(\ell) \setminus \{f'\}$; therefore ℓ cannot accommodate f alongside $\mu^E(\ell)$. This implies the existence of a family $\tilde{f} \in \mu^E(\ell)$ such that $\mu^*(\tilde{f}) \neq \ell$. In fact, because μ^* is obtained by carrying out exactly one Pareto-improving chain, there exists exactly one such family $\tilde{f}$. If $\tilde{f} \rhd_\ell f'$, then the fact that ℓ points at f' implies that $\tilde{f}$ is permanently matched to ℓ at the start of Round i. Following the argument in Sub-case 2.1, we must therefore have that $\mu^*(\tilde{f}) = \mu^E(\tilde{f}) = \ell$, a contradiction. Therefore, we have that $f' \rhd_\ell \tilde{f}$ or $f' = f$. Since $f', \tilde{f} \in \mu^E(\ell)$, needs are monotonic and priorities are lexicographic, by definition we have that $\nu_s^{f'} \geq \nu_s^{\tilde{f}}$ for all $s \in S$. Therefore, the fact that ℓ cannot accommodate f alongside $\mu^E(\ell) \setminus \{f'\}$ implies that ℓ cannot accommodate f alongside $\mu^E(\ell) \setminus \{\tilde{f}\}$, a contradiction. We conclude that $\mu^*(f) \neq \ell$ for all $\mu^* \in M^*$.

We can now conclude that our induction hypothesis holds at the beginning of Round $i+1$. By induction, if a locality ℓ permanently rejects a family f at some point in the MTTCE algorithm, then, for all $\mu^* \in M^*$, $\mu^*(f) \neq \ell$. Therefore, every family matched to its endowment at the end of the algorithm is matched to its endowment under any matching $\mu^* \in M^*$. By assumption, all families are matched to their respective endowments at the end of the MTTCE algorithm, meaning that M^* does not contain any matching other than μ^E. As $\mu^E \notin M^*$ by definition, we conclude that $M^* = \varnothing$ as desired. □

Proof of Proposition 4

Suppose that needs are monotonic and the priority profile is aligned. We introduce the Sequential Deferred Acceptance (SDA) algorithm (Algorithm 6) and show that it produces a stable matching. We show that the alignment of the priority profile allows us to divide the families into groups such that (i) all families in the same group have the same needs and (ii) any two groups can be compared in terms of priority in the sense that all families in one group have a higher priority at all localities than all families in the other group. The SDA algorithm considers one of these groups at a time in order of priority and runs the family-proposing Deferred Acceptance (DA) algorithm for the families in that group, considering the capacities that remain after families in higher-priority groups have been permanently matched.

Claim 1. *The Sequential Deferred Acceptance algorithm produces a stable matching μ^{SDA}.*

First, if there is a directed cycle $(f_1, f_2, \ldots, f_n)$ in graph $\mathbb{G}^1$, then, for every $\ell \in L \setminus \{\emptyset\}$, $f_1 \rhd_\ell f_2 \rhd_\ell \ldots \rhd_\ell f_n \rhd_\ell f_1$, a contradiction. Therefore, $\mathbb{G}^1$ is an acyclic directed graph. By construction, for all $i > 1$, $\mathbb{G}^i$ is an directed acyclic graph for since $\mathbb{G}^i$ is constructed from $\mathbb{G}^{i-1}$ by removing some vertices (families) and edges. Therefore, in every Round i, the set of families $\widetilde{F}^i$ at which no family is pointing is nonempty. Therefore, at least one family gets permanently matched in every round; hence the algorithm ends after at most $|F|$ rounds.

Second, suppose that in Round i, there exist $f, f' \in \widetilde{F}^i$ such that $\nu^f \neq \nu^{f'}$. Since the priority profile is aligned, either $f \rhd_\ell f'$ for all $\ell \in L \setminus \{\emptyset\}$ or $f' \rhd_\ell f$ for all $\ell \in L \setminus \{\emptyset\}$. Therefore, either f points at f', in which case $f' \notin \widetilde{F}^i$, or f' points at f, in which case $f \notin \widetilde{F}^i$, a contradiction. We conclude that in every Round i we have that $\nu^f = \nu^{f'}$ for all families $f, f' \in \widetilde{F}^i$.

We now show that the matching μ^{SDA} produced by the SDA algorithm is stable. Consider any family f and any locality ℓ such that $\ell \succ_f \mu^{SDA}(f)$. We need to show that ℓ cannot accommodate f alongside $\widehat{F}^f_\ell \cap \mu^{SDA}(\ell)$.

Suppose that family f is permanently matched to $\mu^{SDA}(f)$ in Round i, i.e., $f \in \widetilde{F}^i$. Consider the Round i "submarket", in which the DA algorithm permanently matches the families in $\widetilde{F}^i$ to the localities and the counter of each locality ℓ is c^i_ℓ. By construction, all families in the Round i submarket have the same needs, i.e., $\nu^{f'} = \nu^f$ for all $f' \in \widetilde{F}^i$. Therefore, for each locality, there is a maximum number of families that the locality can accommodate, which makes the submarket isomorphic to a school choice problem. As the DA algorithm produces a stable matching in the school choice problem (Abdulkadiroğlu and Sönmez, 2003), the DA algorithm produces a stable matching in the Round i submarket. Therefore, f and ℓ do not form a blocking pair in the Round i submarket. As $\ell \succ_f \mu^{SDA}(f)$ by assumption, it follows that ℓ's Round i counter does not allow ℓ to accommodate f alongside higher-priority families permanently matched to ℓ in Round i, i.e., alongside $\widehat{F}^f_\ell \cap \widetilde{F}^i_\ell$. By construction, all families that have been permanently matched to ℓ before Round i, i.e., all families in $\cup_{j=1}^{i-1} \widetilde{F}^j$, have a higher priority at ℓ than f. Hence, ℓ cannot accommodate f alongside all families that have a higher priority than f and with which ℓ has been permanently matched by the end of Round i, i.e., ℓ cannot accommodate f alongside $\widehat{F}^f_\ell \cap (\cup_{j=1}^{i} \widetilde{F}^j_\ell)$. By construction, all families that are permanently matched to ℓ in any round of the SDA algorithm are matched to ℓ at μ^{SDA}, which implies that $\cup_{j=1}^{i}\widetilde{F}_\ell^j \subseteq \mu^{SDA}(\ell)$. We therefore conclude that ℓ cannot accommodate f alongside $\widehat{F}_\ell^f \cap \mu^{SDA}(\ell)$, as required. □

Proof of Proposition 5

Consider the following modification of the CMDA algorithm. In each round, every locality ℓ tentatively accepts a proposing family f if ℓ can *accommodate* (instead of *weakly accommodate*) f alongside all families with a higher priority than f at ℓ that are proposing to ℓ or have been permanently rejected by ℓ. To see that this modified CMDA algorithm produces the family-optimal envy-free matching, follow the proof of Theorem 5 verbatim having replaced "weakly accommodate" with "accommodate". □

Proof of Proposition 6

Let μ be a matching that is non-wasteful and not stable. We need to show that μ is not weakly envy-free. As μ is not stable, there exists a family f and a locality ℓ such that $\ell \succ_f \mu(f)$ and ℓ can accommodate f alongside $\widehat{F}_\ell^f \cap \mu(\ell)$, that is, for all $s \in S$,

$$\nu_s^f + \sum_{g \in (\widehat{F}_\ell^f \cap \mu(\ell))} \nu_s^g \leq \kappa_s^\ell. \tag{1}$$

However, as μ is non-wasteful, there exists $s \in S$ such that

$$\nu_s^f + \sum_{g \in \mu(\ell)} \nu_s^g > \kappa_s^\ell. \tag{2}$$

Consider now the families in $\mu(\ell) \setminus \widehat{F}_\ell^f$ that require at least one unit of service s. Equations (1) and (2) imply that the set of such families is nonempty. Let f' be the lowest-priority family in $\mu(\ell) \setminus \widehat{F}_\ell^f$ that requires at least one unit of service s (equivalently, f' is the lowest-priority family in $\mu(\ell)$ that requires at least one unit of service s). Then,

$$\nu_s^f + \nu_s^{f'} + \sum_{g \in (\widehat{F}_\ell^{f'} \cap \mu(\ell))} \nu_s^g = \nu_s^f + \sum_{g \in \mu(\ell)} \nu_s^g > \kappa_s^\ell. \tag{3}$$

Inequality (3) and the fact that $\nu_s^{f'} > 0$ imply that ℓ cannot weakly accommodate f' alongside $\{f\} \cup (\widehat{F}_\ell^{f'} \cap \mu(\ell))$. Note that all families in $\{f\} \cup (\widehat{F}_\ell^{f'} \cap \mu(\ell))$ have a higher priority at ℓ than f' and weakly prefer ℓ to the locality to which they are matched, i.e.,

$$(\{f\} \cup (\widehat{F}_\ell^{f'} \cap \mu(\ell))) \subseteq \{g \in F : g \rhd_\ell f' \text{ and } \ell \succeq_g \mu(g)\},$$

so ℓ cannot weakly accommodate f' alongside $\{g \in F : g \rhd_\ell f' \text{ and } \ell \succeq_g \mu(g)\}$. Since $\ell \succ_f \mu(f)$ and $f \rhd_\ell f'$, f strongly envies f' and μ is not weakly envy-free. □

Proof of Theorem 5

We first verify that the output of the CMDA algorithm does not violate any capacity constraint; that is, we show that, for all $\ell \in L$, ℓ can accommodate $\mu^{CMDA}(\ell)$. Consider a locality $\ell \in L$. By construction, all families in $\mu^{CMDA}(\ell)$ propose to and are tentatively accepted by ℓ in the last round of the CMDA algorithm. Therefore, ℓ can weakly accommodate every family $f \in \mu^{CMDA}(\ell)$ alongside $\mu^{CMDA}(\ell) \cap \widehat{F}^f_\ell$. Towards a contradiction, suppose that ℓ cannot accommodate $\mu^{CMDA}(\ell)$. Then, there exists $s \in S$ such that $\sum_{f\in\mu^{CMDA}(\ell)} \nu^f_s > \kappa^\ell_s$. Let $g \in F$ be the lowest-priority family such that $g \in \mu^{CMDA}(\ell)$ and $\nu^g_s > 0$. (Such a family exists since $\sum_{f\in\mu^{CMDA}(\ell)} \nu^f_s > \kappa^\ell_s \geq 0$.) Then,

$$\nu^g_s + \sum_{f\in\mu^{CMDA}(\ell)\cap\widehat{F}^g_\ell} \nu^f_s > \kappa^\ell_s \quad \text{and} \quad \nu^g_s > 0;$$

therefore, ℓ cannot weakly accommodate g alongside $\mu^{CMDA}(\ell) \cap \widehat{F}^g_\ell$ (see Definition 8), a contradiction.

We next show that μ^{CMDA} is weakly envy-free; towards a contradiction, suppose the contrary. Then, by definition, there exists a family f' matched to a locality ℓ' ($\mu^{CMDA}(f') = \ell'$) such that ℓ' cannot weakly accommodate f' alongside $\widehat{G} = \{g \in G : g \rhd_{\ell'} f' \text{ and } \ell' \succeq_g \mu^{CMDA}(g)\}$. Suppose that the CMDA algorithm terminates in Round N. Let $\widehat{R}^N_{f'}(\ell')$ denote the set of families that have a higher priority than f' at ℓ' and propose in Round N to either ℓ' or a less-preferred locality (because they have already been permanently rejected by ℓ' in a previous round). Since all families are matched to the locality to which they propose in Round N, $\widehat{R}^N_{f'}(\ell')$ is also the set of families that have a higher priority than f' at ℓ' and are matched to either ℓ' or to less-preferred localities at the end of the algorithm; therefore $\widehat{R}^N_{f'}(\ell') = \widehat{G}$. Since ℓ' tentatively accepts f' in Round N, ℓ' can weakly accommodate f' alongside $\widehat{R}^N_{f'}(\ell') = \widehat{G}$, a contradiction.

We finally show that μ^{CMDA} dominates all other weakly envy-free matchings. Towards a contradiction, suppose that there exists a weakly envy-free matching μ such that $\mu(f_1) \succ_{f_1} \mu^{CMDA}(f_1)$ for some $f_1 \in F$. We proceed by induction. Our assumption implies that $\mu(f_1)$ permanently rejects f_1 in some Round i_1 of the CMDA algorithm. For the induction argument, suppose that for some $n \in \mathbb{Z}_{>0}$, there exists a family f_n such that $\mu(f_n)$ permanently rejects f_n in some Round i_n of the CMDA algorithm. We want to show that there exists a family f_{n+1} such that $\mu(f_{n+1})$ permanently rejects f_{n+1} in some Round $i_{n+1} < i_n$. By construction, $\mu(f_n)$ cannot weakly accommodate f_n alongside $\widehat{R}^{i_n}_{f_n}(\mu(f_n))$ and all families in $\widehat{R}^{i_n}_{f_n}(\mu(f_n))$ are either matched to $\mu(f_n)$ or a less-preferred locality at μ^{CMDA}. Suppose that, at μ, all families in $\widehat{R}^{i_n}_{f_n}(\mu(f_n))$ are either matched to $\mu(f_n)$ or a less-preferred locality. Then, μ cannot be weakly envy-free since $\mu(f_n)$ cannot weakly accommodate f_n alongside $\widehat{R}^{i_n}_{f_n}(\mu(f_n))$.[29] Therefore, there exists a family $f_{n+1} \in \widehat{R}^{i_n}_{f_n}(\mu(f_n))$ such that $\mu(f_{n+1}) \succ_{f_{n+1}} \mu(f_n)$. Since f_{n+1} proposes to $\mu(f_n)$ in some Round $j \leq i_n$ of the CMDA algorithm, $\mu(f_{n+1})$ permanently rejects f_{n+1} in some Round $i_{n+1} < i_n$. Iterating this argument inductively, we find that some family f_m is permanently rejected by $\mu(f_m)$ in

[29] Recall that part (iii) of Definition 9 implies parts (i) and (ii) of Definition 9.

Round i_m such that $i_m < 1$, impossible. □

Proof of Theorem 6

We first verify that the output of the TMDA mechanism does not violate any capacity constraint; that is, for all $\ell \in L$, ℓ can accommodate $\mu^{TMDA}(\ell)$. Consider a locality $\ell \in L$. By construction, all families in $\mu^{TMDA}(\ell)$ propose to and are tentatively accepted by ℓ in the last round of the TMDA algorithm. Hence, for any $f \in \mu^{TMDA}(\ell)$,

$$\theta_\ell^f \geq |\mu^{TMDA}(\ell) \cap \widehat{F}_\ell^f| + 1 \geq 1.$$

If ℓ cannot weakly accommodate f alongside $\mu^{TMDA}(\ell) \cap \widehat{F}_\ell^f$, then by construction (see Algorithm 5) $\theta_\ell^f = 0$, which contradicts the fact that $\theta_\ell^f \geq 1$. Therefore, ℓ can weakly accommodate every family $f \in \mu^{TMDA}(\ell)$ alongside $\mu^{TMDA}(\ell) \cap \widehat{F}_\ell^f$. As we showed in the proof of Theorem 5, this implies that ℓ can accommodate $\mu^{TMDA}(\ell)$.

We now prove that the TMDA mechanism is strategy-proof and weakly envy-free.

TMDA is strategy-proof

Consider a locality $\ell \in L$ and a subset of families $G \subseteq F$. To simplify notation, let us define, for every $f \in F$, $\widehat{G}_\ell^f = G \cap \widehat{F}_\ell^f$ to be the families in G that have a higher priority for ℓ than f. We also denote by $\theta_\ell^f(G)$ the threshold of family f at locality ℓ if, in some round of the TMDA algorithm, families in G propose to ℓ. (That is, $\theta_\ell^f(G)$ is obtained by running the Threshold Calculator defined in Algorithm 5 with $\Pi_\ell = G$.)

We define the *choice function* of locality ℓ, $C_\ell : 2^F \to 2^F$, as follows: for every $G \subseteq F$, $C_\ell(G) = \{f \in G : |\widehat{G}_\ell^f| + 1 \leq \theta_\ell^f(G)\}$. That is, $C_\ell(G)$ contains the families that ℓ does *not* permanently reject if families in G propose to ℓ in some round of the TMDA algorithm. One way to interpret the choice function is that locality ℓ "chooses" the families in $C_\ell(G)$ when it receives proposals from all families in G.

We now fix a locality $\ell \in L$ and two subsets of families $G \subseteq H \subseteq F$ and define two properties of the choice function:

- *Substitutability (S)*: $C_\ell(H) \cap G \subseteq C_\ell(G)$, and
- *Cardinal Monotonicity (CM)*: $|C_\ell(G)| \leq |C_\ell(H)|$.

Hatfield and Milgrom (2005) analyze properties of the Deferred Acceptance (DA) algorithm, in which localities ("hospitals" in their terminology) have choice functions. In each round of the DA algorithm, families propose to their most preferred locality that has not permanently rejected them yet. Localities tentatively accept or permanently reject proposals based on their choice functions, i.e., if locality ℓ receives proposals from families in G, families in $C_\ell(G)$ are tentatively accepted and families in $G \setminus C_\ell(G)$ are permanently rejected.

Theorems 3 and 11 of Hatfield and Milgrom (2005) imply that the DA algorithm is strategy-proof for families if the choice function of all localities satisfy the (S) and (CM) conditions. By construction, the TMDA algorithm in our setting corresponds to deferred acceptance in the Hatfield and Milgrom (2005) setting with the choice function we have just defined. Therefore, in order to show that the TMDA algorithm is strategy-proof, it remains to prove that the choice function of every locality satisfies the (S) and (CM) conditions.

The following lemma shows three properties of the choice function, which we subsequently use to prove that the choice function satisfies the (S) and (CM) conditions.

Lemma 1. *For every $G \subseteq H \subseteq F$, every $f, g \in F$, and every $\ell \in L$:*

(i) $\theta_\ell^f(G) = \infty$ *if and only if* $\theta_\ell^f(H) = \infty$,

(ii) If $\theta_\ell^f(H) \in \mathbb{Z}_{>0}$, *then* $\theta_\ell^f(G) \leq \theta_\ell^f(H) \leq \theta_\ell^f(G) + |\widehat{H}_\ell^f| - |\widehat{G}_\ell^f|$, *and*

(iii) If $g \triangleright_\ell f$ *and* $\theta_\ell^f(G) \neq \infty$, *then*

- $\theta_\ell^g(G) \geq \theta_\ell^f(G)$, *and*
- $f \in C_\ell(G)$ *implies that* $g \in G \Leftrightarrow g \in C_\ell(G)$.

Part (i) of Lemma 1 states that if the threshold of family f at ℓ is infinite, then it remains so no matter which families are proposing to ℓ. Part (ii) states that if the threshold of family f at ℓ is non-zero and finite, then removing some families from the set of families proposing to ℓ may reduce f's threshold at ℓ by at most the number of proposing families that were removed. Part (iii) states that if the threshold of family f at ℓ is finite, then any family g with a higher priority than f at ℓ has a weakly larger threshold than f and is chosen by ℓ whenever f is chosen by ℓ.

Proof of (S) Consider any family $f \in C_\ell(H) \cap G$; we need to show that $f \in C_\ell(G)$. The fact that $f \in C_\ell(H)$ implies that $\theta_\ell^f(H) \neq 0$ and, if $\theta_\ell^f(H) = \infty$, Lemma 1(i) implies that $\theta_\ell^f(G) = \infty$, hence $f \in C_\ell(G)$. It remains to show that $f \in C_\ell(H)$ whenever $\theta_\ell^f(H) \in \mathbb{Z}_{>0}$. In that case, using Lemma 1(ii), we have that

$$\theta_\ell^f(H) \leq \theta_\ell^f(G) + |\widehat{H}_\ell^f| - |\widehat{G}_\ell^f|.$$

Since $f \in C_\ell(G)$, the definition of a choice function implies that

$$\theta_\ell^f(H) \geq |\widehat{H}_\ell^f| + 1.$$

Combining the two inequalities yields $\theta_\ell^f(G) \geq |\widehat{G}_\ell^f| + 1$ so $f \in C_\ell(G)$, as required.

Proof of (CM) We need to show that $|C_\ell(G)| \leq |C_\ell(H)|$. Let $m = |H \setminus G|$ and arbitrarily label the families in $H \setminus G$ such that $H = G \cup \{f_1, \ldots, f_m\}$.

Claim 2. *For every* $i = 1, \ldots, m$, $|C_\ell(G \cup \{f_1, \ldots, f_{i-1}\})| \leq |C_\ell(G \cup \{f_1, \ldots, f_{i-1}, f_i\})|$.

Claim 2 implies that

$$|C_\ell(G)| \leq |C_\ell(G \cup \{f_1\})| \leq |C_\ell(G \cup \{f_1, f_2\})| \leq \ldots \leq |C_\ell(G \cup \{f_1, \ldots, f_m\})| = |C_\ell(H)|,$$

which implies the desired result. Therefore, it remains to prove Claim 2.

Fix some $i = 1, \ldots, m$ and define $G' = G \cup \{f_1, \ldots, f_{i-1}\}$ and $H' = \{f_1, \ldots, f_{i-1}, f_i\}$. We need to show that $|C_\ell(G')| \le |C_\ell(H')|$. We have that

$$|C_\ell(H')| - |C_\ell(G')| = |C_\ell(H') \setminus C_\ell(G')| - |C_\ell(G') \setminus C_\ell(H')|,$$

so we need to show that

$$|C_\ell(H') \setminus C_\ell(G')| \ge |C_\ell(G') \setminus C_\ell(H')|. \tag{4}$$

By definition, $H' = G' \cup \{f_i\}$ and, by (S), $C_\ell(H') \cap G' \subseteq C_\ell(G')$; therefore,

$$C_\ell(H') \setminus C_\ell(G') \subseteq \{f_i\}.$$

There are two cases: $f_i \notin C_\ell(H')$ and $f_i \in C_\ell(H')$.

Case 1: $f_i \notin C_\ell(H')$. In this case, $C_\ell(H') \setminus C_\ell(G') = \varnothing$ so, by inequality (4), we need to show that $C_\ell(G') \setminus C_\ell(H') = \varnothing$. Towards a contradiction, suppose to the contrary that there exists a family $g \in C_\ell(G') \setminus C_\ell(H')$. Since $g \in C_\ell(G')$, by the definition of the choice function, we have that

$$\theta_\ell^g(G') \ge |\widehat{G}_\ell'^g| + 1, \tag{5}$$

but, since $g \notin C_\ell(H')$, we also have that

$$\theta_\ell^g(H') < |\widehat{H}_\ell'^g| + 1. \tag{6}$$

We now consider two subcases: $g \rhd_\ell f_i$ and $f_i \rhd_\ell g$.

Sub-case 1.1: $g \rhd_\ell f_i$. In this case, as $H' = G' \cup \{f_i\}$, $\widehat{G}_\ell'^g = \widehat{H}_\ell'^g$ so $|\widehat{G}_\ell'^g| = |\widehat{H}_\ell'^g|$ and, as a family's threshold only depends on higher-priority families (Algorithm 5), $\theta_\ell^g(G') = \theta_\ell^g(H')$. Combining these observations with inequalities (5) and (6) yields

$$\theta_\ell^g(G') \ge |\widehat{G}_\ell'^g| + 1 = |\widehat{H}_\ell'^g| + 1 > \theta_\ell^g(H') = \theta_\ell^g(G'),$$

a contradiction.

Sub-case 1.2: $f_i \rhd_\ell g$. On the one hand, inequality (6) implies that $\theta_\ell^g(H') \ne \infty$ so we can apply Lemma 1(i) to obtain that $\theta_\ell^g(G') \ne \infty$ and Lemma 1(iii) to obtain that

$$\theta_\ell^{f_i}(G') \ge \theta_\ell^g(G'). \tag{7}$$

On the other hand, by the assumption that $f_i \notin C_\ell(H')$ in Case 1, we have that

$$\theta_\ell^{f_i}(H') < |\widehat{H}_\ell'^{f_i}| + 1. \tag{8}$$

As $H' = G' \cup \{f_i\}$, $\widehat{G}_\ell'^{f_i} = \widehat{H}_\ell'^{f_i}$ so $|\widehat{G}_\ell'^{f_i}| = |\widehat{H}_\ell'^{f_i}|$ and, as a family's threshold only depends on higher-priority families (Algorithm 5), $\theta_\ell^{f_i}(G') = \theta_\ell^{f_i}(H')$. Combining these observations with inequality (8) yields

$$\theta_\ell^{f_i}(G') = \theta_\ell^{f_i}(H') < |\widehat{H}_\ell'^{f_i}| + 1 = |\widehat{G}_\ell'^{f_i}| + 1. \tag{9}$$

Combining inequality (9) with the fact that $|\widehat{G}_\ell'^{f_i}| \leq |\widehat{G}_\ell'^{g}|$ (as $f_i \triangleright g$) and inequality (5) yields

$$\theta_\ell^{f_i}(G') < |\widehat{G}_\ell'^{f_i}| + 1 \leq |\widehat{G}_\ell'^{g}| + 1 \leq \theta_\ell^{g}(G').$$

We conclude that $\theta_\ell^{f_i}(G') < \theta_\ell^{g}(G')$, which contradicts inequality (7).

Case 2: $f_i \in C_\ell(H')$. In this case, we have that $C_\ell(H') \setminus C_\ell(G') = \{f_i\}$; hence, by inequality (4), we need to show that $|C_\ell(G') \setminus C_\ell(H')| \leq 1$. Towards a contradiction, suppose that there exist two distinct families $g_1, g_2 \in C_\ell(G') \setminus C_\ell(H')$. Without loss of generality, we assume that $g_1 \triangleright_\ell g_2$. Since $g_1, g_2 \in C_\ell(G')$, by the definition of the choice function, we have that

$$\theta_\ell^{g_1}(G') \geq |\widehat{G}_\ell'^{g_1}| + 1 \quad \text{and} \quad \theta_\ell^{g_2}(G') \geq |\widehat{G}_\ell'^{g_2}| + 1, \tag{10}$$

but, since $g_1, g_2 \notin C_\ell(H')$, we also have that

$$\theta_\ell^{g_1}(H') < |\widehat{H}_\ell'^{g_1}| + 1 \quad \text{and} \quad \theta_\ell^{g_2}(H') < |\widehat{H}_\ell'^{g_2}| + 1. \tag{11}$$

We now consider two subcases: $g_1 \triangleright_\ell f_i$ and $f_i \triangleright_\ell g_1$.

Sub-case 2.1: $g_1 \triangleright_\ell f_i$. In this case, as $H' = G' \cup \{f_i\}$, $\widehat{G}_\ell'^{g_1} = \widehat{H}_\ell'^{g_1}$ so $|\widehat{G}_\ell'^{g_1}| = |\widehat{H}_\ell'^{g_1}|$ and, as a family's threshold only depends on higher-priority families (Algorithm 5), $\theta_\ell^{g_1}(G') = \theta_\ell^{g_1}(H')$. Combining these observations with inequalities (10) and (11) yields

$$\theta_\ell^{g_1}(G') \geq |\widehat{G}_\ell'^{g_1}| + 1 = |\widehat{H}_\ell'^{g_1}| + 1 > \theta_\ell^{g_1}(H') = \theta_\ell^{g_1}(G'),$$

a contradiction.

Sub-case 2.2: $f_i \triangleright_\ell g_1$. Inequality (11) implies that $\theta_\ell^{g_2}(H') \neq \infty$ so we can apply Lemma 1(i) to obtain that $\theta_\ell^{g_2}(G') \neq \infty$ and Lemma 1(iii) to obtain that $\theta_\ell^{g_1}(G') \geq \theta_\ell^{g_2}(G')$. Moreover, as $g_1 \triangleright_\ell g_2$ and $g_1 \in G$, we have that $\widehat{G}_\ell^{g_1} \subset \widehat{G}_\ell^{g_2}$ so $|\widehat{G}_\ell^{g_1}| < |\widehat{G}_\ell^{g_2}|$. Combining theses observations with inequality (10) yields

$$\theta_\ell^{g_1}(G') \geq \theta_\ell^{g_2}(G') \geq |\widehat{G}_\ell'^{g_2}| + 1 > |\widehat{G}_\ell'^{g_1}| + 1.$$

We therefore conclude that

$$\theta_\ell^{g_1}(G') \geq |\widehat{G}_\ell'^{g_1}| + 2. \tag{12}$$

Since $\theta_\ell^{g_1}(H') \neq \infty$, we have two cases to consider: $\theta_\ell^{g_1}(H') = 0$ and $\theta_\ell^{g_1}(H') \in \mathbb{Z}_{>0}$.

Sub-sub-case 2.2.1: $\theta_\ell^{g_1}(H') = 0$. In this case, by the definition of thresholds (Algorithm 5), there exists a family $h \in F$ such that (i) either $h = g_1$ or $h \triangleright_\ell g_1$ and (ii) ℓ cannot weakly accommodate h alongside $\widehat{H}_\ell'^{h}$. First, as h has a weakly higher priority than g_1 and $\theta_\ell^{g_1}(G') \neq \infty$ (by inequality (11) and Lemma 1(i)), we can apply Lemma 1(iii) to obtain that $\theta_\ell^{g_1}(G') \leq \theta_\ell^{h}(G')$. Second, as $G' \subseteq H'$, we have that $\widehat{G}_\ell'^{h} \subseteq \widehat{H}_\ell'^{h}$; therefore, $\widehat{H}_\ell'^{h}$ is a subset of $\widehat{F}_\ell^{h}$ that contains all families in $\widehat{G}_\ell'^{h}$ and alongside which ℓ cannot weakly accommodate h. By the definition of thresholds (Algorithm 5), it follows that $\theta_\ell^{h}(G') \leq |\widehat{H}_\ell'^{h}|$. Third, as $H' = G' \cup \{f_i\}$, we have that $|\widehat{H}_\ell'^{h}| \leq |\widehat{G}_\ell'^{h}| + 1$. Fourth, as h has a weakly higher priority than g_1, we have that $|\widehat{G}_\ell'^{h}| \leq |\widehat{G}_\ell'^{g_1}|$. Combining these four observations yields

$$\theta_\ell^{g_1}(G') \leq \theta_\ell^{h}(G') \leq |\widehat{H}_\ell'^{h}| \leq |\widehat{G}_\ell'^{h}| + 1 \leq |\widehat{G}_\ell'^{g_1}| + 1,$$

which contradicts inequality (12).

Sub-sub-case 2.2.2: $\theta_\ell^{g_1}(H') \in \mathbb{Z}_{>0}$. In this case, we can apply Lemma 1(ii) to obtain that $\theta_\ell^{g_1}(H') \geq \theta_\ell^{g_1}(G')$. Moreover, as $H' = G' \cup \{f_i\}$ and $f_i \in \widehat{H}_\ell'^{g_1}$ (by the assumption of Sub-case 2.2), $f_i \rhd_\ell g_1$), we have that $|\widehat{H}_\ell'^{g_1}| = |\widehat{G}_\ell'^{g_1}| + 1$. Combining these observations with inequality (12) yields

$$\theta_\ell^{g_1}(H') \geq \theta_\ell^{g_1}(G') \geq |G_\ell'^{g_1}| + 2 = |H_\ell'^{g_1}| + 1,$$

which contradicts inequality (11) and completes the proof of Claim 2.

TMDA is Weakly Envy-free

Towards a contradiction, suppose that μ^{TMDA} is not weakly envy-free. Then, by Definition 9(iii), there exists a family $f' \in F$ matched to a locality $\ell' \in L$ (i.e., $\mu^{TMDA}(f') = \ell'$) such that ℓ' cannot weakly accommodate f' alongside $\widehat{G} = \{g \in F : g \rhd_{\ell'} f' \text{ and } \ell' \succeq_g \mu^{TMDA}(g)\}$. Let N be the last round of the TMDA algorithm and, for all $i = 1, \ldots, N$, denote by $\Pi_{\ell'}^i$ the set of families that propose to ℓ' and by $\theta_{\ell'}^{f'}(\Pi_{\ell'}^i)$ the threshold of family f' for locality ℓ' in Round i.

Consider Round N. By construction, ℓ' does not permanently reject any family and $\mu^{TMDA}(\ell') = \Pi_{\ell'}^N$. If $\theta_{\ell'}^{f'}(\Pi_{\ell'}^N) = \infty$, then locality ℓ' can weakly accommodate f' alongside $\widehat{F}_{\ell'}^{f'}$. This is a contradiction since $\widehat{G} \subseteq \widehat{F}_{\ell'}^{f'}$. Therefore, $\theta_{\ell'}^{f'}(\Pi_{\ell'}^N) \neq \infty$.

Suppose now that, throughout the algorithm, ℓ' does not permanently reject any family that has a higher priority than f' (i.e., ℓ' does not permanently reject any family in $\widehat{F}_{\ell'}^{f'}$). In that case, all families in $\widehat{F}_{\ell'}^{f'}$ are matched to either ℓ' or a more preferred locality so $\widehat{G} = \Pi_{\ell'}^N \cap \widehat{F}_{\ell'}^{f'}$. Since ℓ' does not permanently reject f', ℓ' can weakly accommodate f' alongside $\widehat{G} = \Pi_{\ell'}^N \cap \widehat{F}_{\ell'}^{f'}$, a contradiction.

We have therefore established that $\theta_{\ell'}^{f'}(\Pi_{\ell'}^N) \neq \infty$ and that locality ℓ' permanently rejects at least one family that has a higher priority than f'. Let g be the highest-priority family that ℓ' permanently rejects (by assumption, $g \rhd_{\ell'} f'$) and let $i = 1, \ldots, N-1$ be the round in which ℓ' permanently rejects g. All families that have a higher priority than g and that propose to ℓ' at some point in the algorithm must also propose to ℓ' in Round N so $(\Pi_{\ell'}^i \cap \widehat{F}_{\ell'}^g) \subseteq (\Pi_{\ell'}^N \cap \widehat{F}_{\ell'}^g)$.

There are two remaining cases to consider: $\theta_{\ell'}^g(\Pi_{\ell'}^i) = 0$ and $\theta_{\ell'}^g(\Pi_{\ell'}^i) \in \mathbb{Z}_{>0}$.

Case 1: $\theta_{\ell'}^g(\Pi_{\ell'}^i) = 0$. By definition, locality ℓ' cannot weakly accommodate g alongside $\Pi_{\ell'}^i \cap \widehat{F}_{\ell'}^g$; therefore ℓ' cannot weakly accommodate g alongside $\Pi_{\ell'}^N \cap \widehat{F}_{\ell'}^g$, meaning that $\theta_{\ell'}^g(\Pi_{\ell'}^N) = 0$. By Lemma 1(iii), $\theta_{\ell'}^{f'}(\Pi_{\ell'}^N) = 0$ so ℓ' permanently rejects f' in Round N, a contradiction.

Case 2: $\theta_{\ell'}^g(\Pi_{\ell'}^i) \in \mathbb{Z}_{>0}$. By assumption, locality ℓ' permanently rejects g in Round i. First, since $(\Pi_{\ell'}^i \cap \widehat{F}_{\ell'}^g) \subseteq (\Pi_{\ell'}^N \cap \widehat{F}_{\ell'}^g)$, Lemma 1(ii) applies and yields

$$\theta_{\ell'}^g(\Pi_{\ell'}^N \cap \widehat{F}_{\ell'}^g) \leq \theta_{\ell'}^g(\Pi_{\ell'}^i \cap \widehat{F}_{\ell'}^g) + |\Pi_{\ell'}^N \cap \widehat{F}_{\ell'}^g| - |\Pi_{\ell'}^i \cap \widehat{F}_{\ell'}^g|. \tag{13}$$

Second, from Algorithm 5, the threshold of a family at a given locality only depends on families with a higher priority at that locality; therefore

$$\theta^g_{\ell'}(\Pi^i_{\ell'}) = \theta^g_{\ell'}(\Pi^i_{\ell'} \cap \widehat{F}^g_{\ell'}) \quad \text{and} \quad \theta^g_{\ell'}(\Pi^N_{\ell'}) = \theta^g_{\ell'}(\Pi^N_{\ell'} \cap \widehat{F}^g_{\ell'}). \tag{14}$$

Finally, by Lemma 1(iii), we have that

$$\theta^{f'}_{\ell'}(\Pi^N_{\ell'}) \leq \theta^g_{\ell'}(\Pi^N_{\ell'}). \tag{15}$$

Combining (13), (14), and (15), we obtain that

$$\theta^{f'}_{\ell'}(\Pi^N_{\ell'}) \leq \theta^g_{\ell'}(\Pi^N_{\ell'}) \leq \theta^g_{\ell'}(\Pi^i_{\ell'}) + |\Pi^N_{\ell'} \cap \widehat{F}^g_{\ell'}| - |\Pi^i_{\ell'} \cap \widehat{F}^g_{\ell'}|. \tag{16}$$

Finally, the fact that $g \rhd_{\ell'} f'$ implies that $|\Pi^N_{\ell'} \cap \widehat{F}^g_{\ell'}| \leq |\Pi^N_{\ell'} \cap \widehat{F}^{f'}_{\ell'}|$ and the fact that ℓ' permanently rejects g in Round i implies that $\theta^g_{\ell'}(\Pi^i_{\ell'}) < |\Pi^i_{\ell'} \cap \widehat{F}^g_{\ell'}| + 1$, or, equivalently, that $\theta^g_{\ell'}(\Pi^i_{\ell'}) - |\Pi^i_{\ell'} \cap \widehat{F}^g_{\ell'}| < 1$. Therefore, we have that

$$\theta^g_{\ell'}(\Pi^i) + |\Pi^N_{\ell'} \cap \widehat{F}^g_{\ell'}| - |\Pi^i_{\ell'} \cap \widehat{F}^g_{\ell'}| < |\Pi^N_{\ell'} \cap \widehat{F}^{f'}_{\ell'}| + 1. \tag{17}$$

Combining (16) and (17) yields $\theta^{f'}_{\ell'}(\Pi^N_{\ell'}) < |\Pi^N_{\ell'} \cap \widehat{F}^{f'}_{\ell'}| + 1$. Therefore, ℓ' permanently rejects f' in Round N, a contradiction. □

Proof of Lemma 1

Proof of (i) If $\theta^f_\ell(G) = \infty$, then ℓ can weakly accommodate f alongside $\widehat{F}^f_\ell$, which implies $\theta^f_\ell(H) = \infty$. The converse is proved analogously.

Proof of (ii) We have assumed that $\theta^f_\ell(H) \in \mathbb{Z}_{>0}$. We first show that $\widetilde{\theta}^f_\ell(H), \widetilde{\theta}^f_\ell(G), \theta^f_\ell(G) \in \mathbb{Z}_{>0}$. If either $\widetilde{\theta}^f_\ell(H) = \infty$, $\widetilde{\theta}^f_\ell(G) = \infty$, or $\theta^f_\ell(G) = \infty$, ℓ can weakly accommodate f alongside $\widehat{F}^f_\ell$, which implies that $\theta^f_\ell(H) = \infty$, a contradiction. It remains to show that none of $\widetilde{\theta}^f_\ell(H)$, $\widetilde{\theta}^f_\ell(G)$, or $\theta^f_\ell(G)$ are equal to 0. Since $\widetilde{\theta}^f_\ell(H) \neq \infty$, $\theta^f_\ell(H) \leq \widetilde{\theta}^f_\ell(H)$ by definition (Algorithm 5); therefore $\widetilde{\theta}^f_\ell(H) = 0$ implies $\theta^f_\ell(H) = 0$, a contradiction. If $\widetilde{\theta}^f_\ell(G) = 0$, then ℓ cannot weakly accommodate f alongside $\widehat{G}^f_\ell$. Since $\widehat{G}^f_\ell \subseteq \widehat{H}^f_\ell$, ℓ cannot weakly accommodate f alongside $\widehat{H}^f_\ell$ so $\widetilde{\theta}^f_\ell(H) = 0$, a contradiction. If $\theta^f_\ell(G) = 0$ and $\widetilde{\theta}^f_\ell(G) \neq 0$, then there exists $g \in \widehat{F}^f_\ell$ such that $\widetilde{\theta}^g_\ell(G) = 0$. Since $G \subseteq H$, we have that $\widetilde{\theta}^g_\ell(H) = 0$. Hence, as $g \in \widehat{F}^f_\ell$, we have that $\theta^f_\ell(H) = 0$, a contradiction.

Having established that $\widetilde{\theta}^f_\ell(H), \widetilde{\theta}^f_\ell(G), \theta^f_\ell(G) \in \mathbb{Z}_{>0}$, we next show that $\widetilde{\theta}^f_\ell(G) \leq \widetilde{\theta}^f_\ell(H)$. As $\widetilde{\theta}^f_\ell(H) \in \mathbb{Z}_{>0}$, there exists a subset of families $\widetilde{F}$ such that $\widehat{H}^f_\ell \subseteq \widetilde{F} \subseteq \widehat{F}^f_\ell$ and $|\widetilde{F}| = \widetilde{\theta}^f_\ell(H)$ alongside which ℓ cannot weakly accommodate f. Since $G \subseteq H$, we have that $\widehat{G}^f_\ell \subseteq \widehat{H}^f_\ell$ and therefore $\widehat{G}^f_\ell \subseteq \widehat{H}^f_\ell \subseteq \widetilde{F}$. Hence, we obtain that $\widetilde{\theta}^f_\ell(G) \leq |\widetilde{F}| = \widetilde{\theta}^f_\ell(H)$ as required.

We next show that $\widetilde{\theta}^f_\ell(H) \leq \widetilde{\theta}^f_\ell(G) + |\widehat{H}^f_\ell| - |\widehat{G}^f_\ell|$. As $\widetilde{\theta}^f_\ell(G) \in \mathbb{Z}_{>0}$, there exists a subset of families $\widetilde{F}$ such that $\widehat{G}^f_\ell \subseteq \widetilde{F} \subseteq \widehat{F}^f_\ell$ and $|\widetilde{F}| = \widetilde{\theta}^f_\ell(G)$ alongside which ℓ cannot weakly accommodate f. Therefore, ℓ cannot weakly accommodate f alongside $\widetilde{F} \cup \widehat{H}^f_\ell$ so $\widetilde{\theta}^f_\ell(H) \leq |\widetilde{F} \cup \widehat{H}^f_\ell|$. By construction, $|\widetilde{F} \cup \widehat{H}^f_\ell| = |\widetilde{F}| + |\widehat{H}^f_\ell| - |\widetilde{F} \cap \widehat{H}^f_\ell|$ and $\widehat{G}^f_\ell \subseteq (\widetilde{F} \cap \widehat{H}^f_\ell)$

so $|\widehat{G}^f_\ell| \leq |\widetilde{F} \cap \widehat{H}^f_\ell|$. Combining the preceding results with the fact that $|\widetilde{F}| = \widetilde{\theta}^f_\ell(G)$ yields

$$\widetilde{\theta}^f_\ell(H) \leq |\widetilde{F} \cup \widehat{H}^f_\ell| = |\widetilde{F}| + |\widehat{H}^f_\ell| - |\widetilde{F} \cap \widehat{H}^f_\ell| \leq |\widetilde{F}| + |\widehat{H}^f_\ell| - |\widehat{G}^f_\ell| = \widetilde{\theta}^f_\ell(G) + |\widehat{H}^f_\ell| - |\widehat{G}^f_\ell|.$$

We have now established that $\widetilde{\theta}^f_\ell(G) \leq \widetilde{\theta}^f_\ell(H) \leq \widetilde{\theta}^f_\ell(G) + |\widehat{H}^f_\ell| - |\widehat{G}^f_\ell|$. Since f was chosen arbitrarily, we have

$$\min_{g \in \widehat{F}^f_\ell \cup \{f\}} \widetilde{\theta}^g_\ell(G) \leq \min_{g \in \widehat{F}^f_\ell \cup \{f\}} \widetilde{\theta}^g_\ell(H) \leq \min_{g \in \widehat{F}^f_\ell \cup \{f\}} \left\{ \widetilde{\theta}^g_\ell(G) + |\widehat{H}^g_\ell| - |\widehat{G}^g_\ell| \right\}. \tag{18}$$

Let $h = \arg\min_{g \in \widehat{F}^f_\ell \cup \{f\}} \widetilde{\theta}^g_\ell(G)$. Then, we have that

$$\min_{g \in \widehat{F}^f_\ell \cup \{f\}} \left\{ \widetilde{\theta}^g_\ell(G) + |\widehat{H}^g_\ell| - |\widehat{G}^g_\ell| \right\} \leq \widetilde{\theta}^h_\ell(G) + |\widehat{H}^h_\ell| - |\widehat{G}^h_\ell|. \tag{19}$$

In addition, we have that

$$|\widehat{H}^h_\ell| - |\widehat{G}^h_\ell| = |\widehat{H}^h_\ell \setminus \widehat{G}^h_\ell| \leq |\widehat{H}^f_\ell \setminus \widehat{G}^f_\ell| = |\widehat{H}^f_\ell| - |\widehat{G}^f_\ell|, \tag{20}$$

where the two equalities follow from the fact that $\widehat{G}^h_\ell \subseteq \widehat{H}^h_\ell$ and $\widehat{G}^f_\ell \subseteq \widehat{H}^f_\ell$ while the fact that $(\widehat{H}^h_\ell \setminus \widehat{G}^h_\ell) \subseteq (\widehat{H}^f_\ell \setminus \widehat{G}^f_\ell)$ (as $h \in \widehat{F}^f_\ell$) implies the inequality. Combining (18), (19), and (20), we obtain

$$\min_{g \in \widehat{F}^f_\ell \cup \{f\}} \widetilde{\theta}^g_\ell(G) \leq \min_{g \in \widehat{F}^f_\ell \cup \{f\}} \widetilde{\theta}^g_\ell(H) \leq \widetilde{\theta}^h_\ell(G) + |\widehat{H}^f_\ell| - |\widehat{G}^f_\ell|. \tag{21}$$

Finally, using the fact that $\theta^f_\ell(G), \theta^f_\ell(H) \in \mathbb{Z}_{>0}$ and the definition of θ^f_ℓ from Algorithm 5, we obtain that

$$\theta^f_\ell(G) = \min_{g \in \widehat{F}^f_\ell \cup \{f\}} \widetilde{\theta}^g_\ell(G) = \widetilde{\theta}^h_\ell(G) \quad \text{and} \quad \theta^f_\ell(H) = \min_{g \in \widehat{F}^f_\ell \cup \{f\}} \widetilde{\theta}^g_\ell(H). \tag{22}$$

Combining (21) and (22) yields

$$\theta^f_\ell(G) \leq \theta^f_\ell(H) \leq \theta^f_\ell(G) + |\widehat{H}^f_\ell| - |\widehat{G}^f_\ell|,$$

as required.

Proof of (iii) We have assumed that $g \rhd_\ell f$ and $\theta^f_\ell(G) \neq \infty$. To prove the first part of the statement, we need to show that $\theta^g_\ell(G) \geq \theta^f_\ell(G)$. If $\theta^g_\ell(G) = \infty$, the result is immediate since $\theta^f_\ell(G)$ is finite. If $\theta^g_\ell(G) \neq \infty$, the definition of thresholds in Algorithm 5 implies that

$$\theta^f_\ell(G) = \min_{h \in \widehat{G}^f_\ell} \widetilde{\theta}^h_\ell \quad \text{and} \quad \theta^g_\ell(G) = \min_{h \in \widehat{G}^g_\ell} \widetilde{\theta}^h_\ell.$$

Then, the fact that $g \rhd_\ell f$ implies $\widehat{G}_\ell^g \subseteq \widehat{G}_\ell^f$; therefore

$$\theta_\ell^g(G) = \min_{h \in \widehat{G}_\ell^g} \widetilde{\theta}_\ell^h \geq \min_{h \in \widehat{G}_\ell^f} \widetilde{\theta}_\ell^h = \theta_\ell^f(G),$$

as required.

To prove the second part of the statement we need to show that $g \in G \Leftrightarrow g \in C_\ell(G)$ under the additional assumption that $f \in C_\ell(G)$. By definition of a choice function, if $g \in C_\ell(G)$, then $g \in G$. It remains to show that $f \in C_\ell(G)$ and $g \in G$ imply that $g \in C_\ell(G)$.

By definition of a choice function, $g \in C_\ell(G)$ whenever $\theta_\ell^g(G) \geq |\widehat{G}_\ell^g| + 1$ or, equivalently, whenever $\theta_\ell^g(G) - |\widehat{G}_\ell^g| \geq 1$. We have already established in the first part of the statement that $\theta_\ell^g(G) \geq \theta_\ell^f(G)$. Additionally, since $g \rhd_\ell f$ and $g \in G$, we have that $\widehat{G}_\ell^g \subset \widehat{G}_\ell^f$, and therefore $|\widehat{G}_\ell^g| < |\widehat{G}_\ell^f|$. Taken together, these results imply that

$$\theta_\ell^g(G) - |\widehat{G}_\ell^g| > \theta_\ell^f(G) - |\widehat{G}_\ell^f|. \tag{23}$$

Since $f \in C_\ell(G)$ by assumption, we have that $\theta_\ell^f(G) \geq |\widehat{G}_\ell^f| + 1$ (by definition of a choice function). Combining $\theta_\ell^f(G) - |\widehat{G}_\ell^f| \geq 1$ with (23) yields

$$\theta_\ell^g(G) - |\widehat{G}_\ell^g| > \theta_\ell^f(G) - |\widehat{G}_\ell^f| \geq 1.$$

We can conclude that $\theta_\ell^g(G) > |\widehat{G}_\ell^g| + 1$. By definition of a choice function, we have that $g \in C_\ell(G)$, which completes the proof. □

Online Appendix

A Efficiency of the TMDA algorithm

A.1 Lower bound on the efficiency of CMDA and TMDA mechanisms

We now show the lower bound on the efficiency of the CMDA and TMDA mechanisms.

Proposition 8. *The CMDA and TMDA mechanisms match at least one family to a locality in $L \setminus \{\emptyset\}$. If all families can be accommodated on their own at all localities, then the CMDA and TMDA mechanisms match at least* $\min\{|F|, |L|-1\}$ *families to localities in $L \setminus \{\emptyset\}$.*

The intuition for Proposition 8 is as follows. First, if a family f that can be accommodated at some locality ℓ is matched to the null locality, then f has been permanently rejected by ℓ. Therefore, there exists a family f' with a higher priority than f at ℓ that (i) can be accommodated on its own at ℓ and (ii) proposes to ℓ in some round of the CMDA or TMDA algorithm. In turn, family f' would only be matched to the null locality if there is a yet another higher priority family f'' and, by induction, it is not possible that all families be matched to the null locality. Second, consider the case where all families can be accommodated on their own at all localities. If fewer than $|L|-1$ families are matched to non-null localities, then either there are fewer than $|L|-1$ families in the market or at least one family f is matched to the null locality while a non-null locality ℓ is not matched to any family. This yields a contradiction since f proposes to and is permanently rejected by ℓ in some round of the CMDA or TMDA algorithms.

A.2 TMDA with Clinching algorithm

In this section, we present a modification of the TMDA algorithm that improves its efficiency without affecting its properties. The TMDA with Clinching (TMDAC) algorithm (Algorithm 7) starts with a Clinching Round that creates a new priority profile. The TMDA algorithm is then run using the new priority profile.

Proposition 9. *The TMDAC mechanism is strategy-proof and weakly envy-free. Moreover,* $\mu^{TMDAC} \succeq \mu^{TMDA}$.

The idea of the TMDAC algorithm is to identify family-locality pairs that will necessarily be matched together by the TMDA algorithm. If locality ℓ is family f's first preference and ℓ can weakly accommodate f alongside all higher-priority families, then $\mu^{TMDA}(f) = \ell$: f proposes to ℓ and ℓ tentatively accepts f's proposal throughout the algorithm because f's threshold at ℓ is ∞. In the Clinching Round of the TMDAC algorithm, family f *clinches* locality ℓ. Then we construct a new priority profile in which f moves to the bottom of the priority list of every locality ℓ' such that $\ell \succ_f \ell'$. The change in the priority profile does not affect f's match in the TMDA algorithm, since f will be matched to ℓ, no matter what f's priority for less-preferred localities. However, the change in the priority profile may positively affect families that propose to f's less-preferred localities since their thresholds at these localities are no longer affected by f.

		Round 1			Round 2			Round 3			Round 4		
f_1	$\to$	ℓ_1	[1]	✓	ℓ_1	[0]	✗	ℓ_2	[∞]	✓	ℓ_2	[∞]	✓
f_2	$\to$	ℓ_2	[1]	✓	ℓ_2	[1]	✓	ℓ_2	[0]	✗	ℓ_1	[∞]	✓
f_3	$\to$	ℓ_2	[1]	✗	ℓ_1	[∞]	✓	ℓ_1	[∞]	✓	ℓ_1	[∞]	✓

Table 4: TMDAC* algorithm applied to Example 4.

To illustrate how clinching can improve families' welfare, consider a locality ℓ_1 and three families f_1, f_2, and f_3 such that $\rhd_{\ell_1} : f_1, f_2, f_3, \ldots$. There is only one service s_1 and the needs and capacities are

$$\begin{array}{cccccc} & f_1 & f_2 & f_3 & \ell_1 & \\ s_1 & (\; 2 & 1 & 1 & 2 \;). \end{array}$$

Consider what happens if f_2 and f_3 both propose to ℓ_1 in some round of the TMDA algorithm. The thresholds are $\theta_{\ell_1}^{f_1} = \infty$ (since f_1 has the highest priority for ℓ_1 and ℓ_1 can accommodate f_1 on its own) and $\theta_{\ell_1}^{f_2} = \theta_{\ell_1}^{f_3} = 1$ (since ℓ_1 can (weakly) accommodate each of f_2 or f_3 on its own but not alongside f_1). Therefore, ℓ_1 tentatively accepts f_2's proposal (since f_2's priority rank among proposing families is $1 = \theta_{\ell_1}^{f_2}$) and permanently rejects f_3's proposal (since f_3's priority rank among proposing families is $2 > 1 = \theta_{\ell_1}^{f_3}$). Observe that ℓ_1 permanently rejects f_3 even though ℓ_1 can (weakly) accommodate both proposing families.

Suppose, however, that f_1's first preference is another locality ℓ_2 and that ℓ_2 can (weakly) accommodate f_1 alongside all higher-priority families. Then, the clinching round identifies this pair and f_1 clinches ℓ_2. The Clinching Round produces a priority profile $\widetilde{\rhd}$ such that $f_2 \widetilde{\rhd}_{\ell_1} f_3 \widetilde{\rhd}_{\ell_1} f_1$. If the TMDA algorithm is run with the new priority profile $\widetilde{\rhd}$ and f_2 and f_3 both propose to ℓ_1, we have that $\theta_{\ell_1}^{f_2} = \theta_{\ell_1}^{f_3} = \infty$ since f_2 and f_3 now have the two highest priorities for ℓ_1 and therefore ℓ_1 can accommodate both f_2 and f_3. As a consequence, following the Clinching Round, ℓ_1 no longer permanently rejects f_3 resulting into a Pareto improvement over the original TMDA mechanism.

The TMDAC algorithm also allows families to clinch localities that are not their first preferences. In Step 1 of the Clinching Round, locality ℓ rejects family f if ℓ cannot accommodate f on its own (as no clinches have occurred before the start of Step 1). If ℓ rejects f and f receives a proposal from its second-preference locality ℓ', i.e., if ℓ' can weakly accommodate f alongside all higher-priority families, then it can be established that the TMDA algorithm would match f to ℓ'. Therefore, f clinches ℓ', i.e., f goes to the bottom of the priority list of all of f's less-preferred localities. In Step 2, these localities may propose to new families as a result. In addition, ℓ' now rejects every family with a lower priority than f at ℓ' that ℓ' cannot weakly accommodate alongside f. The Clinching Round continues until there is a step in which no family clinches any locality. Online Appendix B.4 provides an example of the TMDAC algorithm.

A.3 Why clinching any proposing locality affects incentives for truth-telling

As clinching improves efficiency, one might consider allowing families to clinch any locality that proposes to them (whether or not they have been rejected by all of their more preferred localities). Suppose we allowed family f to clinch a locality ℓ as long as ℓ can weakly

Algorithm 7: TMDA WITH CLINCHES (TMDAC)

Clinching Round

Step 0

No locality rejects or proposes to any family and no family clinches any locality.

Set $\widetilde{\rhd}^1 = \rhd$ and continue to Step 1.

Step $j \geq 1$

(a) Locality ℓ rejects family f if ℓ cannot weakly accommodate f alongside the families that (i) clinched ℓ in Step $j-1$ and (ii) are higher than f on $\widetilde{\rhd}^j_\ell$.

(b) Locality ℓ proposes to family f if ℓ can weakly accommodate f alongside all families that are higher than f on $\widetilde{\rhd}^j_\ell$.

(c) Family f clinches locality ℓ if (i) ℓ proposed to f in part (b) and (ii) ℓ is f's most preferred locality that did not reject f in part (a).

(d) If at least one clinch occurred in part (c) that did not occur in Step $j-1$, continue to part (e). Otherwise, set $\widetilde{\rhd} = \widetilde{\rhd}^j$ and continue to the TMDA algorithm.

(e) Construct $\widetilde{\rhd}^{j+1}$ as follows, then continue to Step $j+1$. For every $\ell \in L$ and every $f, f' \in F$ with $f \rhd_\ell f'$,

- $f' \widetilde{\rhd}^{j+1}_\ell f$ if, in part (c), (i) f clinched a locality that f strictly prefers to ℓ and (ii) f' did not clinch a locality that f' strictly prefers to ℓ;
- $f \widetilde{\rhd}^{j+1}_\ell f'$ otherwise.

TMDA

Run the TMDA algorithm with the priority profile $\widetilde{\rhd}$.

accommodate f alongside all higher-priority families, even if there are other localities that f prefers and that have not rejected f.

Formally, consider the following modification to the Clinching Round in Algorithm 7: remove condition (ii) of part (c) in every Step j. We call TMDAC* algorithm the TMDAC algorithm with the modified Clinching Round. Therefore, the TMDAC* algorithms differs from the TMDAC algorithm by the fact that, in the Clinching Round, a family f clinches *any* locality that proposes to f.

Unfortunately, the TMDAC* algorithm is not strategy-proof, as the following example shows.

Example 4. There are three families, three localities, and one service. The preferences, priorities, service needs, and service capacities are

$$\begin{array}{llllll} \succ_{f_1}: & \ell_1, \ell_2, \ell_3 & \succ_{f_2}: & \ell_2, \ell_1, \ell_3 & \succ_{f_3}: & \ell_1, \ell_2, \ell_3 \\ \rhd_{\ell_1}: & f_2, f_3, f_1 & \rhd_{\ell_2}: & f_1, f_2, f_3 & \rhd_{\ell_3}: & f_1, f_2, f_3 \end{array}$$

$$\nu = \begin{array}{c} \\ s_1 \end{array} \begin{array}{c} \begin{matrix} f_1 & f_2 & f_3 \end{matrix} \\ \begin{pmatrix} 2 & 1 & 1 \end{pmatrix} \end{array} \qquad \kappa = \begin{array}{c} \\ s_1 \end{array} \begin{array}{c} \begin{matrix} \ell_1 & \ell_2 & \ell_3 \end{matrix} \\ \begin{pmatrix} 2 & 2 & 2 \end{pmatrix} \end{array}.$$

Suppose that all families report truthfully. In the modified Clinching Round, every family receives a proposal from its second-preference locality. Since ℓ_3 is every families' third- and last-preference locality, the modified Clinching Round does not affect the localities' priorities. As a result, the priority profile $\widetilde{\rhd}$ that is used in the TMDA algorithm is the same as the original priority profile $\rhd$.[30]

The TMDA algorithm—summarized in Table 4—yields the following matching:

$$\begin{pmatrix} f_1 & f_2 & f_3 \\ \ell_2 & \ell_1 & \ell_1 \end{pmatrix}.$$

Suppose now that family f_1 reports $\succ'_{f_1}: \ell_1, \ell_3, \ell_2$ while the other two families report their true preferences. In the modified Clinching Round, f_1 clinches ℓ_3 and drops to the bottom of ℓ_2's priority list; therefore, we have a new priority list $\widetilde{\rhd}_{\ell_2} : f_2, f_3, f_1$.[31] Families f_2 and f_3 clinch locality f_1, but this proposal is inconsequential since f_3 already has the highest priority for ℓ_3. Therefore, the modified Clinching Round does not modify the priorities of ℓ_1 and ℓ_3, i.e., $\widetilde{\rhd}_{\ell_1} = \rhd_{\ell_1}$ and $\widetilde{\rhd}_{\ell_3} = \rhd_{\ell_3}$.

The TMDA algorithm is then run with the priority profile $\widetilde{\rhd}$. In the first round, f_1 proposes to ℓ_1 and is tentatively accepted as $\theta^{f_1}_{\ell_1} = 1$ and no other family proposes to ℓ_1. Families f_2 and f_3 both propose to ℓ_2. Since f_2 and f_3 both now have a higher priority than f_1 and ℓ_2 can accommodate them together and ℓ_2 tentatively accepts both families (as $\theta^{f_2}_{\ell_2} = \theta^{f_3}_{\ell_2} = \infty$). The TMDA algorithm ends and yields the matching

$$\begin{pmatrix} f_1 & f_2 & f_3 \\ \ell_1 & \ell_2 & \ell_2 \end{pmatrix}.$$

Clearly, f_1's manipulation has been successful since f_1 is now matched to its first-preference locality ℓ_1 instead of its second-preferences locality ℓ_2. The reason why f can successfully manipulate is that by clinching ℓ_3, f_1 allows f_3 to be tentatively accepted by ℓ_2; as a result, f_3 does not compete with f_1 for ℓ_1.

Note that the TMDAC algorithm precludes f_1's manipulation opportunity in Example 4 because the TMDAC algorithm only allows f_1 to clinch a ℓ_3 when it has been established that the TMDA algorithm will not match f_1 to any locality that f_1 prefers to ℓ_3 (i.e., ℓ_1).

[30] Family f_1 also receives a proposal from ℓ_3, but this proposal is inconsequential since ℓ_3 is f_1's least-preferred locality.

[31] Family f_1 also receives a proposal from ℓ_2, but this proposal is inconsequential since ℓ_2 is f_1's reported least-preferred locality.

A.4 Proofs

Proof of Proposition 8

We first show that the TMDA mechanism matches at least one family to a locality. Recall that we assume throughout that for every family $f \in F$, there exists a locality $\ell \in L \setminus \{\emptyset\}$ such that ℓ can accommodate f on its own (see Section 3). Without loss of generality, let f be the highest-priority family among those that ℓ can accommodate on their own. Recall that we assume throughout that localities prioritize families they can accommodate on their own over families that they cannot accommodate (see Section 3) so f has the highest priority at ℓ among *all* families. Therefore, $\theta_\ell^f = \infty$ throughout the TMDA algorithm, which means that ℓ does not permanently reject f in any round of the TMDA algorithm. If f proposes to ℓ, $\mu^{TMDA}(f) = \ell \succ_f \emptyset$; otherwise, f does not propose to ℓ, so $\mu^{TMDA}(f) \succ_f \ell \succ_f \emptyset$. In both cases, f is matched to a locality that is not the null.

We next show that if all families can be accommodated on their own at all localities then the TMDA mechanism matches at least $\min\{|F|, |L|-1\}$ families to a locality other than the null. Towards a contradiction, suppose that $|F \setminus \mu^{TMDA}(\emptyset)| < \min\{|F|, |L|-1\}$. Then, $\mu^{TMDA}(\emptyset) \neq \varnothing$ and there exists $\ell \in L \setminus \{\emptyset\}$ such that $\mu^{TMDA}(\ell) = \varnothing$. Since we assume that $\emptyset$ is every family's last preference and $\mu^{TMDA}(\ell) = \varnothing$, ℓ receives at least one proposal. Let f be the highest-priority family among those that propose to ℓ at least once in the TMDA algorithm. Since the hypothesis states that ℓ can accommodate f on its own, $\theta_\ell^f \geq 1$ throughout the TMDA algorithm. In every round in which f proposes to ℓ, f has the highest priority among proposing families so ℓ never permanently rejects f. This means that $f \in \mu^{TMDA}(\ell)$, which contradicts $\mu^{TMDA}(\ell) = \varnothing$.

Since the CMDA mechanism is family optimal, we have that $\mu^{CMDA} \succeq \mu^{TMDA}$. Moreover, $\emptyset$ is every family's last preference. Hence, the result also holds for the CMDA mechanism. □

Proof of Proposition 9

First notice that the TMDAC algorithm simply runs the TMDA algorithm with a different priority profile. Since the matching produced by the TMDA algorithm does not violate any priority constraint (see the proof of Theorem 6), the same is true of the TMDAC algorithm, i.e., for every $\ell \in L$, ℓ can accommodate $\mu^{TMDAC}(\ell)$.

We next introduce some notation, which we use throughout the proof. Consider the Clinching Round and let N be its total number of steps. We use throughout the convention that $\widetilde{\triangleright}^0 = \triangleright$. For every Step $j = 0, 1, \ldots, N$, every $f \in F$, and every $\ell \in L$, let

- $\widehat{F}_\ell^f(\widetilde{\triangleright}^j)$ be the set of families that are higher than f on $\widetilde{\triangleright}_\ell^j$;
- Δ_f^j be the set of localities that reject f in Step j;
- Γ_f^j be the set of localities that propose to f in Step j; and
- Θ_ℓ^j be the set of families that clinch ℓ in Step j.

Note that $\widehat{F}_\ell^f(\widetilde{\triangleright}^0) = \widehat{F}_\ell^f(\widetilde{\triangleright}^1) = \widehat{F}_\ell^f(\triangleright) = \widehat{F}_\ell^f$ and $\widehat{F}_\ell^f(\widetilde{\triangleright}^N) = \widehat{F}_\ell^f(\widetilde{\triangleright})$. Since no rejections, proposals, or clinches occur in Step 0, we also have $\Delta_f^0 = \Gamma_f^0 = \Theta_\ell^0 = \varnothing$.

The following two lemmata are key to our analysis. Their proofs can be found directly after the current proof.

Lemma 2. *For every Step $j = 1, \dots, N$, every family $f \in F$, and every locality $\ell \in L$:*

(i) If $f \notin \Theta_{\ell'}^{j-1}$ for all $\ell' \in L$ such that $\ell' \succ_f \ell$, then $\widehat{F}_\ell^f(\widetilde{\triangleright}^j) \subseteq \widehat{F}_\ell^f(\widetilde{\triangleright}^{j-1})$;

(ii) $\Delta_f^{j-1} \subseteq \Delta_f^j$;

(iii) If $\ell \in \Gamma_f^{j-1} \setminus \Gamma_f^j$, then there exists $\ell' \in L$ such that $\ell' \succ_f \ell$ and $f \in \Theta_{\ell'}^{j-1}$; and

(iv) $\Theta_\ell^{j-1} \subseteq \Theta_\ell^j$.

Part (i) of Lemma 2 says that if family f has not clinched a locality that f prefer to ℓ, then the set of families who are higher than f at ℓ shrinks throughout the Clinching Round. Part (ii) says that the set of localities that are rejecting f grows throughout the Clinching Round. Part (iii) says that if ℓ stops proposing to f, then f has clinched a more preferred locality. Part (iv) says that the set of families that have cliched ℓ grows throughout the Clinching Round.

Lemma 3. *For every Step $j = 1, \dots, N$, every family $f \in F$, and every locality $\ell \in L$:*

(i) If $\ell \in \Delta_f^j$, then $\mu^{TMDAC}(f) \neq \ell$;

(ii) If $\ell \in \Gamma_f^j$, then $\mu^{TMDAC}(f) \succeq_f \ell$; and

(iii) If $f \in \Theta_\ell^j$, then $\mu^{TMDAC}(f) = \ell$.

Part (i) of Lemma 3 says that if a locality ℓ rejects a family f in the Clinching Round, then the TMDAC algorithm will not match f to ℓ. Part (ii) says that if ℓ proposes to f in the Clinching Round, then f will be matched to ℓ or a more preferred locality under the TMDAC algorithm. Part (iii) says that if f clinches ℓ in the Clinching Round, then the TMDAC algorithm will match f to ℓ.

We now use Lemmata 2 and 3 to show that the TMDAC mechanism is strategy-proof and weakly envy-free and that $\mu^{TMDAC} \succeq \mu^{TMDA}$.

TMDAC is Strategy-proof

We consider a family f and a report $\succ_f'$. We need to show that

$$\varphi^{TMDAC}(\succ)(f) \succeq \varphi^{TMDAC}(\succ_f', \succ_{-f})(f).$$

We use our usual notation—N, $\widetilde{\triangleright}^j$, Δ_f^j, Γ_f^j, and Θ_ℓ^j—for the TMDAC algorithm run with the preference profile $\succ$, i.e., when f reports truthfully. We denote by $\overline{N}$, $\overline{\widetilde{\triangleright}}^j$, $\overline{\Delta}_f^j$, $\overline{\Gamma}_f^j$, and $\overline{\Theta}_\ell^j$ the counterparts in the TMDAC algorithm run with the preference profile $(\succ_f', \succ_{-f})$, i.e., when f misreports its preferences.

Consider the Clinching Round when the preference profile is $\succ$. If f clinches a locality, let Step m be the first step in which f clinches the locality; formally, $f \notin \cup_{\ell \in L}\{\Theta_\ell^j\}$ for all $j = 1, \ldots, m-1$ and $f \in \cup_{\ell \in L}\{\Theta_\ell^j\}$ for all $j = m, \ldots, N$. (By Lemma 2(iv), once a family clinches a locality, it continues to clinch the same locality in all remaining steps so m is well defined.) If f does not clinch any locality, let $m = \infty$; formally, $m = \infty$ whenever $f \notin \cup_{\ell \in L}\{\Theta_\ell^N\}$. We define $\overline{m}$ analogously for the preference profile $(\succ'_f, \succ_{-f})$. The following lemma guarantees that the Clinching Round is unaffected by f's report until f clinches a locality.

Lemma 4. *Let $q = \min\{m, \overline{m}, N, \overline{N}\}$. For every $j = 1, \ldots, q$, every $g \in F$, and every $\ell \in L$,*

$$\Theta_\ell^j \setminus \{f\} = \overline{\Theta}_\ell^j \setminus \{f\}, \quad \Delta_g^j = \overline{\Delta}_g^j, \quad \Gamma_g^j = \overline{\Gamma}_g^j, \quad \textit{and} \quad \widetilde{\rhd}^j = \overline{\rhd}^j.$$

Moreover, if $j < \min\{m, \overline{m}\}$, then $\Theta_\ell^j = \overline{\Theta}_\ell^j$ for all $\ell \in L$.

(The proof of Lemma 4 follows after the current proof.) There are four cases to consider.

Case 1: $m = \infty$ and $\overline{m} = \infty$. In this case, f does not clinch any locality in the Clinching Round, irrespective of whether f reports $\succ_f$ or $\succ'_f$. Then, $q = \min\{N, \overline{N}\} < \min\{m, \overline{m}\}$. If $N \leq \overline{N}$, then $q = N$ and $\Theta_\ell^{N-1} = \Theta_\ell^N$ for all $\ell \in L$ since, by construction, the Clinching Round ends when the same clinches occur in two consecutive steps. By Lemma 4, $\Theta_\ell^{N-1} = \overline{\Theta}_\ell^{N-1}$ and $\Theta_\ell^N = \overline{\Theta}_\ell^N$ for all $\ell \in L$; therefore, $\overline{\Theta}_\ell^{N-1} = \overline{\Theta}_\ell^N$ for all $\ell \in L$, which means that $N = \overline{N}$. Then, since $\widetilde{\rhd}^N = \overline{\rhd}^N$ by Lemma 4, we conclude that $\widetilde{\rhd} = \overline{\rhd}$. We have established that, whether f reports $\succ_f$ or $\succ'_f$, the Clinching Round ends in the same step and produces the same adjusted priority profile. Then, $\varphi^{TMDAC}(\succ)(f)$ is the matching produced by the TMDA algorithm when the preference and priority profiles are $\succ$ and $\widetilde{\rhd}$ respectively while $\varphi^{TMDAC}(\succ'_f, \succ_{-f})(f)$ is the matching produced by the TMDA algorithm when the preference and priority profiles are $(\succ'_f, \succ_{-f})$ and $\overline{\rhd} = \widetilde{\rhd}$ respectively. Since the TMDA algorithm is strategy-proof (Theorem 6), we conclude that $\varphi^{TMDAC}(\succ)(f) \succeq_f \varphi^{TMDAC}(\succ'_f, \succ_{-f})(f)$, as required. Analogous reasoning yields the same result for the case where $N \geq \overline{N}$.

Case 2: $m \leq \min\{\overline{m}, N\}$. In this case, if f reports truthfully, then f clinches a locality in Step m of the Clinching Round and, if f reports $\succ'_f$, then f either clinches a locality in Step $\overline{m} \geq m$ or does not clinch any locality. Since $m \leq \min\{\overline{m}, N\}$, $q = \min\{m, \overline{N}\}$. Towards a contradiction, suppose that $m > \overline{N}$. Then, $q = \overline{N} < \min\{m, \overline{m}\}$. As the Clinching Round ends whenever the same clinches occur in two consecutive rounds, we have that $\Theta_\ell^{\overline{N}-1} = \Theta_\ell^{\overline{N}}$ for all $\ell \in L$. Moreover, Lemma 4 implies that $\Theta_\ell^{\overline{N}-1} = \overline{\Theta}_\ell^{\overline{N}-1}$ and $\Theta_\ell^{\overline{N}} = \overline{\Theta}_\ell^{\overline{N}}$ for all $\ell \in L$. It follows that $\overline{\Theta}_\ell^{\overline{N}-1} = \overline{\Theta}_\ell^{\overline{N}}$, so $N = \overline{N}$, which contradicts our assumption that $\overline{N} < m \leq N$.

We have established that $m \leq \overline{N}$; hence $q = m$. When f reports truthfully, f clinches a locality denoted by ℓ in Step m of the Clinching Round, i.e., $f \in \Theta_\ell^m$. By Lemma 3(iii), we have that $\varphi^{TMDAC}(\succ)(f) = \ell$. By construction (Step m(c) of the Clinching Round), $f \in \Theta_\ell^m$ implies that, for all $\ell' \in L$ with $\ell' \succ_f \ell$, $\ell' \in \Delta_f^m$. By Lemma 4, $\Delta_f^m = \overline{\Delta}_f^m$; therefore $\ell' \in \overline{\Delta}_f^m$ for all $\ell' \in L$ such that $\ell' \succ_f \ell$. By Lemma 3(i), it follows that $\varphi^{TMDAC}(\succ'_f, \succ_{-f})(f) \neq \ell'$ for all $\ell' \in L$ such that $\ell' \succ_f \ell$; therefore $\ell \succeq_f \varphi^{TMDAC}(\succ'_f, \succ_{-f})(f)$, as required.

Case 3: $\overline{m} \leq \min\{m, \overline{N}\}$. In this case, if f reports $\succ'_f$, then f clinches a locality in Step $\overline{m}$ of the Clinching Round and, if f reports truthfully, then f either clinches a locality in Step $m \geq \overline{m}$ or does not clinch any locality. Since $\overline{m} \leq \{m, \overline{N}\}$, $q = \min\{\overline{m}, N\}$. Using analogous reasoning to Case 2, we establish that $\overline{m} \leq N$; hence $q = \overline{m}$. When f reports $\succ'_f$, f clinches a locality denoted by ℓ in Step $\overline{m}$ of the Clinching Round, i.e., $f \in \overline{\Theta}_\ell^{\overline{m}}$. By Lemma 3(iii), we have that $\varphi^{TMDAC}(\succ'_f, \succ_{-f}) = \ell$. By construction (Step $\overline{m}$(c) of the Clinching Round), $f \in \overline{\Theta}_\ell^{\overline{m}}$ implies that $\ell \in \overline{\Gamma}_f^{\overline{m}}$. By Lemma 4, $\Gamma_f^{\overline{m}} = \overline{\Gamma}_f^{\overline{m}}$ so $\ell \in \Gamma_f^{\overline{m}}$; therefore, by Lemma 3(ii), $\varphi^{TMDAC}(\succ)(f) \succeq_f \ell$, as required.

TMDAC is Weakly Envy-free

For ease of notation, let $\ell = \mu^{TMDAC}(f)$.

Claim 3. *For any family $f \in F$,*

$$\{g \in F : g \rhd_\ell f \text{ and } \ell \succeq_g \mu^{TMDAC}(g)\} \subseteq \{g \in F : g \,\widetilde{\rhd}_\ell f \text{ and } \ell \succeq_g \mu^{TMDAC}(g)\}.$$

Consider a family $f \in F$ and suppose, towards a contradiction, that there exists another family $h \in F$ such that

$$h \in \{g \in F : g \rhd_\ell f \text{ and } \ell \succeq_g \mu^{TMDAC}(g)\} \setminus \{g \in F : g \,\widetilde{\rhd}_\ell f \text{ and } \ell \succeq_g \mu^{TMDAC}(g)\}.$$

By assumption, $h \rhd_\ell f$ and $f \,\widetilde{\rhd}_\ell\, h$; equivalently, $f \in \widehat{F}_\ell^h(\widetilde{\rhd}) \setminus \widehat{F}_\ell^h$. Therefore, there exists a Step $j = 1, \ldots, N$ of the Clinching Round such that $f \in \widehat{F}_\ell^h(\widetilde{\rhd}^j) \setminus \widehat{F}_\ell^h(\widetilde{\rhd}^{j-1})$. By the contrapositive of Lemma 2(i), it follows that $h \in \Theta_{\ell'}^{j-1}$ for some $\ell' \in L$ such that $\ell' \succ_h \ell$. By Lemma 3(iii), we have that $\mu^{TMDAC}(h) = \ell'$. We conclude that $\mu^{TMDAC}(h) \succ_h \ell$, a contradiction since, by assumption, $\ell \succeq_h \mu^{TMDAC}(h)$. This establishes Claim 3.

We now show that μ^{TMDAC} is weakly envy-free. Towards a contradiction, suppose μ^{TMDAC} is not weakly envy-free. Then, there exists a family $f \in F$ such that ℓ cannot weakly accommodate f alongside $\{g \in F : g \rhd_\ell f \text{ and } \ell \succeq_g \mu^{TMDAC}(g)\}$. By Claim 3, it follows that ℓ cannot weakly accommodate f alongside $\{g \in F : g \,\widetilde{\rhd}_\ell f \text{ and } \ell \succeq_g \mu^{TMDAC}(g)\}$. As μ^{TMDAC} is by construction the result of running the TMDA algorithm with the priority profile $\widetilde{\rhd}$, we conclude that the TMDA mechanism is not weakly envy-free, a contradiction of Theorem 6.

$\mu^{TMDAC} \succeq \mu^{TMDA}$

We first introduce some additional notation. Let M be the number of rounds of the TMDA algorithm and, for every $i = 1, \ldots, M$ and every $\ell \in L$, let Π_ℓ^i be the set of families that propose to ℓ in Round i. Similarly, let $\widetilde{M}$ be the number of rounds of the TMDAC algorithm; that is, the TMDAC algorithm consists of a Clinching Round, which lasts N steps, and then the TMDA algorithm is run with the constructed priority profile $\widetilde{\rhd}$ and lasts $\widetilde{M}$ rounds. For every $i = 1, \ldots, \widetilde{M}$ and every $\ell \in L$, let $\widetilde{\Pi}_\ell^i$ be the set of families that propose to ℓ in Round i of the TMDAC algorithm. By definition (Algorithm 5), the threshold of a family f at a locality ℓ only depends on higher-priority families; therefore, $\theta_\ell^f(\Pi_\ell^i \cap \widehat{F}_\ell^f)$ is the threshold of family f for locality ℓ in Round i of the TMDA algorithm and $\theta_\ell^f(\widetilde{\Pi}_\ell^i \cap \widehat{F}_\ell^f(\widetilde{\rhd}))$ is the threshold of family f for locality ℓ in Round i of the TMDAC algorithm.

Claim 4. *In every Round $i = 1, \ldots, \widetilde{M}$ of the TMDAC algorithm, every family $f \in F$ proposes to a locality that f weakly prefers to $\mu^{TMDA}(f)$.*

By construction, for every family $f \in F$, $\mu^{TMDAC}(f)$ is the locality to which f proposes in Round $\widetilde{M}$ of the TMDAC algorithm. Therefore, Claim 4 implies that $\mu^{TMDAC}(f) \succeq_f \mu^{TMDA}(f)$ for all $f \in F$, as required. It remains to prove Claim 4.

In Round 1 of the TMDAC algorithm, every family proposes to its most preferred locality; therefore Claim 4 holds for $i = 1$. The remainder of the proof is by induction. We suppose that Claim 4 holds for some $i = 1, \ldots, \widetilde{M} - 1$ (induction hypothesis) and show that Claim 4 holds for $i + 1$.

Consider any family $f \in F$ and, for ease of notation, let $\ell = \mu^{TMDA}(f)$. We need to show that, in Round $i + 1$ of the TMDAC algorithm, f proposes to a locality that f weakly prefers to ℓ. By our induction hypothesis, f proposes to a locality that f weakly prefers to ℓ in Round i. If f proposes to a locality that f strictly prefers to ℓ in Round i, then ℓ does not permanently reject f in Round i so f proposes to a locality that f weakly prefers to ℓ in Round $i + 1$, as required. We therefore focus on the case where f proposes to ℓ in Round i of the TMDAC algorithm and need to show that ℓ tentatively accepts f's proposal. That is, by the construction of the TMDA part of the TMDAC algorithm, we need to show that

$$\theta_\ell^f(\widetilde{\Pi}_\ell^i \cap \widehat{F}_\ell^f) \geq |\widetilde{\Pi}_\ell^i \cap \widehat{F}_\ell^f| + 1. \tag{24}$$

First, as f proposes to ℓ in Round i, $\ell \succeq_f \mu^{TMDAC}(f)$ by construction. Then, by Lemma 3(iii), f does not clinch any locality that f strictly prefers to ℓ in the Clinching Round. By Lemma 2(i), it follows that $\widehat{F}_\ell^f(\widetilde{\triangleright}) \subseteq \widehat{F}_\ell^f$.

Second, by construction, family f proposes to and is tentatively accepted by ℓ in the last round of the TMDA algorithm; hence

$$\theta_\ell^f(\Pi_\ell^M \cap \widehat{F}_\ell^f) \geq |\Pi_\ell^M \cap \widehat{F}_\ell^f| + 1. \tag{25}$$

There are two cases: $\theta_\ell^f(\Pi^M \cap \widehat{F}_\ell^f) = \infty$ and $\theta_\ell^f(\Pi^M \cap \widehat{F}_\ell^f) \neq \infty$.

Case 1: $\theta_\ell^f(\Pi^M \cap \widehat{F}_\ell^f) = \infty$. In this case, by the definition of thresholds (Algorithm 5), ℓ can weakly accommodate f alongside $\widehat{F}_\ell^f$. As $\widehat{F}_\ell^f(\widetilde{\triangleright}) \subseteq \widehat{F}_\ell^f$, ℓ can weakly accommodate f alongside $\widehat{F}_\ell^f(\widetilde{\triangleright})$. Again, by the definition of thresholds (Algorithm 5), we have that $\theta_\ell^f(\widetilde{\Pi}^i \cap \widehat{F}_\ell^f(\widetilde{\triangleright})) = \infty$; hence inequality (24) is satisfied, as required.

Case 2: $\theta_\ell^f(\Pi^M \cap \widehat{F}_\ell^f) \neq \infty$. There are two sub-cases: ℓ permanently rejects at least one family with a higher priority than f at ℓ in the TMDA algorithm and ℓ does not permanently reject any family with a higher priority than f at ℓ in the TMDA algorithm.

Sub-case 2.1: ℓ permanently rejects at least one family with a higher priority than f at ℓ in the TMDA algorithm. In this case, let $g \in F$ be the highest-priority family that ℓ permanently rejects in the TMDA algorithm. Therefore, there exists a Round $j = 1, \ldots, M$ of the TMDA algorithm, $g \in \Pi_\ell^j$ and

$$\theta_\ell^g(\Pi_\ell^j \cap \widehat{F}_\ell^g) < |\Pi_\ell^j \cap \widehat{F}_\ell^g| + 1. \tag{26}$$

As g is the highest-priority family that ℓ permanently rejects in the TMDA algorithm, any family with a higher priority that proposes to ℓ in Round j continues to propose to ℓ until the end of the algorithm; therefore, $\Pi_\ell^j \cap \widehat{F}_\ell^g \subseteq \Pi_\ell^M \cap \widehat{F}_\ell^g$.

We next show that the following inequality holds:

$$\theta_\ell^g(\Pi_\ell^M \cap \widehat{F}_\ell^g) < |\Pi_\ell^M \cap \widehat{F}_\ell^g| + 1. \tag{27}$$

First, inequality (27) holds trivially if $\theta_\ell^g(\Pi_\ell^M \cap \widehat{F}_\ell^g) = 0$. Second, inequality (26) implies that $\theta_\ell^g(\Pi_\ell^j \cap \widehat{F}_\ell^g) \neq \infty$; hence Lemma 1(i) implies that $\theta_\ell^g(\Pi_\ell^M \cap \widehat{F}_\ell^g) \neq \infty$. Third, if $\theta_\ell^g(\Pi_\ell^M \cap \widehat{F}_\ell^g) \in \mathbb{Z}_{>0}$, then, as $\Pi_\ell^j \cap \widehat{F}_\ell^g \subseteq \Pi_\ell^M \cap \widehat{F}_\ell^g$, we can apply Lemma 1(ii) to obtain that

$$\theta_\ell^g(\Pi_\ell^M \cap \widehat{F}_\ell^g) \leq \theta_\ell^g(\Pi_\ell^j \cap \widehat{F}_\ell^g) + |\Pi_\ell^M \cap \widehat{F}_\ell^g| - |\Pi_\ell^j \cap \widehat{F}_\ell^g|. \tag{28}$$

Combined with inequality (26), inequality (28) implies inequality (27).

As $g \rhd_\ell f$, we have that $\Pi_\ell^M \cap \widehat{F}_\ell^g \subseteq \Pi_\ell^M \cap \widehat{F}_\ell^f$. Moreover, as a family's threshold only depends on higher-priority families (Algorithm 5), $\theta_\ell^g(\Pi_\ell^M \cap \widehat{F}_\ell^f) = \theta_\ell^g(\Pi_\ell^M \cap \widehat{F}_\ell^g)$. Combining these observations with inequalities (27) and (25) yields

$$\theta_\ell^g(\Pi_\ell^M \cap \widehat{F}_\ell^f) = \theta_\ell^g(\Pi_\ell^M \cap \widehat{F}_\ell^g) < |\Pi_\ell^M \cap \widehat{F}_\ell^g| + 1 \leq |\Pi_\ell^M \cap \widehat{F}_\ell^f| + 1 \leq \theta_\ell^f(\Pi_\ell^M \cap \widehat{F}_\ell^f). \tag{29}$$

By inequality (26), $\theta_\ell^g(\Pi_\ell^M \cap \widehat{F}_\ell^f) \neq \infty$; hence $\theta_\ell^g(\Pi_\ell^M \cap \widehat{F}_\ell^g) \neq \infty$. Hence, we can apply Lemma 1(iii) to obtain that

$$\theta_\ell^g(\Pi_\ell^M \cap \widehat{F}_\ell^f) \geq \theta_\ell^f(\Pi_\ell^M \cap \widehat{F}_\ell^f),$$

which contradicts inequality (29). We therefore conclude that Sub-case 2.1 cannot occur.

Sub-case 2.2: ℓ does not permanently reject any family with a higher priority than f at ℓ in the TMDA algorithm. In this case, we first show that $\widetilde{\Pi}_\ell^i \cap \widehat{F}_\ell^f \subseteq \Pi_\ell^M \cap \widehat{F}_\ell^f$. Towards a contradiction, suppose that there exists a family $g \in (\widetilde{\Pi}_\ell^i \cap \widehat{F}_\ell^f) \setminus (\Pi_\ell^M \cap \widehat{F}_\ell^f)$. As $(\widetilde{\Pi}_\ell^i \cap \widehat{F}_\ell^f) \setminus (\Pi_\ell^M \cap \widehat{F}_\ell^f) = (\widetilde{\Pi}_\ell^i \setminus \Pi_\ell^M) \cap \widehat{F}_\ell^f$, we have that $g \in \widetilde{\Pi}_\ell^i$, $g \notin \Pi_\ell^M$, and $g \in \widehat{F}_\ell^f$.

First, by our induction hypothesis, g proposes in Round i of the TMDAC algorithm to a locality that g weakly prefers to $\mu^{TMDA}(g)$; therefore, the fact that $g \in \widetilde{\Pi}_\ell^i$ implies that $\ell \succeq_g \mu^{TMDA}(g)$. Second, as the TMDA algorithm matches g to the last locality to which g proposes, the fact that $g \notin \Pi_\ell^M$ implies that $\ell \neq \mu^{TMDA}(g)$. Third, by the assumption of Sub-case 2.2, the fact that $g \in \widehat{F}_\ell^f$ implies that ℓ does not permanently reject g in the TMDA algorithm, hence $\mu^{TMDA}(g) \succeq_g \ell$. Combining these observations, we have that $\ell \succeq_g \mu^{TMDA}(g)$, $\ell \neq \mu^{TMDA}(g)$, and $\mu^{TMDA}(g) \succeq_g \ell$, a contradiction.

Having established that $\widetilde{\Pi}_\ell^i \cap \widehat{F}_\ell^f \subseteq \Pi_\ell^M \cap \widehat{F}_\ell^f$ and that $\widehat{F}_\ell^f(\widetilde{\rhd}) \subseteq \widehat{F}_\ell^f$, we conclude that $\widetilde{\Pi}_\ell^i \cap \widehat{F}_\ell^f(\widetilde{\rhd}) \subseteq \Pi_\ell^M \cap \widehat{F}_\ell^f$. It follows that $|\widetilde{\Pi}_\ell^i \cap \widehat{F}_\ell^f(\widetilde{\rhd})| \leq |\Pi_\ell^M \cap \widehat{F}_\ell^f|$. Moreover, as $\theta_\ell^f(\Pi_\ell^M \cap \widehat{F}_\ell^f) \neq \infty$ by the assumption of Case 2, and as $\theta_\ell^f(\Pi_\ell^M \cap \widehat{F}_\ell^f) \neq 0$ by inequality (25), we have that $\theta_\ell^f(\Pi_\ell^M \cap \widehat{F}_\ell^f) \in \mathbb{Z}_{>0}$ so we can apply Lemma 1(ii) to obtain that

$$\theta_\ell^f(\Pi_\ell^M \cap \widehat{F}_\ell^f) \leq \theta_\ell^f(\widetilde{\Pi}_\ell^i \cap \widehat{F}_\ell^f(\widetilde{\rhd})) + |\Pi_\ell^M \cap \widehat{F}_\ell^f| - |\widetilde{\Pi}_\ell^i \cap \widehat{F}_\ell^f(\widetilde{\rhd})|. \tag{30}$$

Combining inequality (30) with inequality (25) yields inequality (24), as required. □

Proof of Lemma 2

We prove the lemma by a single induction argument. To show that the lemma holds for $j = 1$, note that (i) $\widetilde{\rhd}^0 = \widetilde{\rhd}^1 = \rhd$,(ii) $\varnothing = \Delta_f^0 \subseteq \Delta_f^1$ for every $f \in F$, (iii) $\varnothing = \Gamma_f^0 \subseteq \Gamma_f^1$ for every $f \in F$, and (iv) $\varnothing = \Theta_\ell^0 \subseteq \Theta_\ell^1$ for every $\ell \in L$.

For the induction step, let us assume that part (iv) of the lemma holds for some $j = 1, \ldots, N$, i.e., $\Theta_\ell^{j-1} \subseteq \Theta_\ell^j$ for every $\ell \in L$ and every $j = 1, \ldots, N$. We will show that the assumption implies parts (i)-(iv) of the lemma in Step $j+1$. That is, we consider an arbitrary family $f \in F$ and an arbitrary locality $\ell \in L$ and show the following:

(i) If $f \notin \Theta_{\ell'}^j$ for all $\ell' \in L$ such that $\ell' \succ_f \ell$, then $\widehat{F}_\ell^f(\widetilde{\rhd}^{j+1}) \subseteq \widehat{F}_\ell^f(\widetilde{\rhd}^j)$;

(ii) $\Delta_f^j \subseteq \Delta_f^{j+1}$;

(iii) If $\ell \in \Gamma_f^j \setminus \Gamma_f^{j+1}$, then there exists $\ell' \in L$ such that $\ell' \succ_f \ell$ and $f \in \Theta_{\ell'}^j$; and

(iv) $\Theta_\ell^j \subseteq \Theta_\ell^{j+1}$.

Proof of (i) We prove the contrapositive. Suppose there exists a family $g \in \widehat{F}_\ell^f(\widetilde{\rhd}^{j+1}) \setminus \widehat{F}_\ell^f(\widetilde{\rhd}^j)$, then $f \, \widetilde{\rhd}_\ell^j g$ and $g \, \widetilde{\rhd}_\ell^{j+1} f$. We need to show that f clinches a locality that f strictly prefers to ℓ in Step j. There are two cases: $f \rhd_\ell g$ and $g \rhd_\ell f$.

Case 1: $f \rhd_\ell g$. Since $g \, \widetilde{\rhd}_\ell^{j+1} f$, then by construction (Step j(e) of the Clinching Round) we have that f clinches a locality that f strictly prefers to ℓ in Step j(c) of the Clinching Round.

Case 2: $g \rhd_\ell f$. Since $f \, \widetilde{\rhd}_\ell^j g$, then by construction (Step j(e) of the Clinching Round), g clinches a locality that g strictly prefers to ℓ in Step $j-1$(c) of the Clinching Round. Since we have assumed that $\Theta_\ell^{j-1} \subseteq \Theta_\ell^j$, g continues to clinch that locality in Step j(c) of the Clinching Round. Since $g \, \widetilde{\rhd}_\ell^{j+1} f$, we have that f clinches a locality that f strictly prefers to ℓ in Step j(c) of the Clinching Round.

Proof of (ii) Suppose that $\ell \in \Delta_f^j$, we need to show that $\ell \in \Delta_f^{j+1}$. As $\ell \in \Delta_f^j$, ℓ cannot weakly accommodate f alongside $\Theta_\ell^{j-1} \cap \widehat{F}_\ell^f(\widetilde{\rhd}^j)$. Similarly, $\ell \in \Delta_f^{j+1}$ if ℓ cannot weakly accommodate f alongside $\Theta_\ell^j \cap \widehat{F}_\ell^f(\widetilde{\rhd}^{j+1})$. Therefore, it is sufficient to show that

$$(\Theta_\ell^{j-1} \cap \widehat{F}_\ell^f(\widetilde{\rhd}^j)) \subseteq (\Theta_\ell^j \cap \widehat{F}_\ell^f(\widetilde{\rhd}^{j+1})).$$

Towards a contradiction, suppose that there exists a family

$$g \in (\Theta_\ell^{j-1} \cap \widehat{F}_\ell^f(\widetilde{\rhd}^j)) \setminus (\Theta_\ell^j \cap \widehat{F}_\ell^f(\widetilde{\rhd}^{j+1})).$$

Since we have assumed that $\Theta_\ell^{j-1} \subseteq \Theta_\ell^j$, we have that

$$g \in (\Theta_\ell^{j-1} \cap \widehat{F}_\ell^f(\widetilde{\rhd}^j)) \setminus \widehat{F}_\ell^f(\widetilde{\rhd}^{j+1}) = \Theta_\ell^{j-1} \cap (\widehat{F}_\ell^f(\widetilde{\rhd}^j) \setminus \widehat{F}_\ell^f(\widetilde{\rhd}^{j+1})).$$

Since $g \in \widehat{F}_\ell^f(\widetilde{\rhd}_\ell^j) \setminus \widehat{F}_\ell^f(\widetilde{\rhd}^{j+1})$, we have that $g \, \widetilde{\rhd}_\ell^j f$ and $f \, \widetilde{\rhd}_\ell^{j+1} g$. Therefore, $f \in \widehat{F}_\ell^g(\widetilde{\rhd}_\ell^{j+1}) \setminus \widehat{F}_\ell^g(\widetilde{\rhd}^j)$ and, as a result, $\widehat{F}_\ell^g(\widetilde{\rhd}_\ell^{j+1}) \not\subseteq \widehat{F}_\ell^g(\widetilde{\rhd}^j)$. Then, the contrapositive of (i) implies that $g \in \Theta_{\ell'}^j$ for some $\ell' \in L$ such that $\ell' \succ_g \ell$. Since $g \in \Theta_\ell^{j-1}$ and we have assumed that $\Theta_\ell^{j-1} \subseteq \Theta_\ell^j$, it follows that g clinches both ℓ and ℓ' in Step j. This is a contradiction since, by construction (Step j(c) of the Clinching Round), a family can clinch at most one locality in any given step of the Clinching Round.

Proof of (iii) Suppose that $\ell \in \Gamma_f^j \setminus \Gamma_f^{j+1}$. Then, ℓ can weakly accommodate f alongside $\widehat{F}_\ell^f(\widetilde{\triangleright}^j)$ but not alongside $\widehat{F}_\ell^f(\widetilde{\triangleright}^{j+1})$, which implies that $\widehat{F}_\ell^f(\widetilde{\triangleright}^{j+1}) \not\subseteq \widehat{F}_\ell^f(\widetilde{\triangleright}^j)$. By the contrapositive of (i), we have that $f \in \Theta_{\ell'}^j$ for some $\ell' \in L$ such that $\ell' \succ_f \ell$.

Proof of (iv) Suppose that $f \in \Theta_\ell^j$, we need to show that $f \in \Theta_\ell^{j+1}$. By construction (Step j(c) of Clinching Round), ℓ proposes to f in Step j(b) and all localities that f prefers to ℓ reject f in Step j(a); that is, $\ell \in \Gamma_f^j$ and, for all $\ell' \in L$ such that $\ell' \succ_f \ell$, $\ell' \in \Delta_f^j$. Since a family can clinch at most one locality in each step, $f \notin \Theta_{\ell'}^j$ for all $\ell' \in L$ such that $\ell' \succ_f \ell$. By the contrapositive of (iii), $\ell \notin \Gamma_f^j \setminus \Gamma_f^{j+1}$; therefore $\ell \in \Gamma_f^{j+1}$. Moreover, by (ii), $\ell' \in \Delta_f^{j+1}$ for all $\ell' \in L$ such that $\ell' \succ_f \ell$. Therefore, ℓ proposes to f in Step $j+1$ and all localities that f prefers to ℓ reject f in Step $j+1$, which means that f clinches ℓ in Step $j+1$, i.e., $f \in \Theta_\ell^{j+1}$. □

Proof of Lemma 3

For ease of notation, let $\mu = \mu^{TMDAC}$. We consider a Step $j = 1, \ldots, N$ of the Clinching Round, a family $f \in F$, and a locality $\ell \in L$. We prove each part of the lemma in turn.

Proof of (i) We first show that if a family g is rejected by $\mu(g)$ in Step $i = 2, \ldots, N$ of the Clinching Round, then a family h is rejected by $\mu(h)$ in Step $i-1$.

Suppose that a family $g \in F$ is rejected by $\mu(g)$ in Step $i = 2, \ldots, N$ of the Clinching Round. Since $\mu(g)$ rejects g in Step i(a) of the Clinching Round, $\mu(g)$ cannot weakly accommodate g alongside $\Theta_{\mu(g)}^{i-1} \cap \widehat{F}_{\mu(g)}^g(\widetilde{\triangleright}^i)$. If all families in $\Theta_{\mu(g)}^{i-1} \cap \widehat{F}_{\mu(g)}^g(\widetilde{\triangleright}^i)$ are matched to $\mu(g)$ at the end of the TMDAC algorithm (i.e., if $(\Theta_{\mu(g)}^{i-1} \cap \widehat{F}_{\mu(g)}^g(\widetilde{\triangleright}^i)) \subseteq \mu(\mu(g))$), then $\mu(g)$ can accommodate $(\Theta_{\mu(g)}^{i-1} \cap \widehat{F}_{\mu(g)}^g(\widetilde{\triangleright}^i)) \cup \{g\}$. Therefore, $\mu(g)$ can weakly accommodate g alongside $\Theta_{\mu(g)}^{i-1} \cap \widehat{F}_{\mu(g)}^g(\widetilde{\triangleright}^i)$, a contradiction. We conclude that there exists a family $h \in \Theta_{\mu(g)}^{i-1} \cap \widehat{F}_{\mu(g)}^g(\widetilde{\triangleright}^i)$ such that $\mu(h) \neq \mu(g)$. Since $h \in \Theta_{\mu(g)}^{i-1}$, we have that $h \in \Theta_{\mu(g)}^N$ by Lemma 2(iv). It follows that $\mu(g)$ proposes to h in Step N of the Clinching Round, which in turn implies that $\mu(g)$ can weakly accommodate h alongside $\widehat{F}_\ell^h(\widetilde{\triangleright}^N) = \widehat{F}_\ell^h(\widetilde{\triangleright})$. Then, in every round of the TMDAC algorithm, h's threshold at $\mu(g)$ is ∞ (see Algorithm 5). It follows that $\mu(g)$ tentatively accepts any proposal from h so the fact that $\mu(h) \neq \mu(g)$ implies that h does not propose to $\mu(g)$ in the TMDAC algorithm, hence $\mu(h) \succ_h \mu(g)$. Finally, since $h \in \Theta_{\mu(g)}^{i-1}$ and $\mu(h) \succ_f \mu(g)$, $\mu(h)$ rejects h in Step $i-1$(c) of the Clinching Round, as required.

Now, suppose towards a contradiction that $\mu(f) \in \Delta_f^j$. The preceding argument implies, by induction, that there exists a family f' such that $\mu(f') \in \Delta_{f'}^1$, i.e., $\mu(f')$ rejects f' in Step 1 of the Clinching Round. Then, $\mu(f')$ cannot weakly accommodate f' alongside $\Theta_{\mu(f')}^0 \cap \widehat{F}_{\mu(f')}^{f'}(\widetilde{\triangleright}^1)$. Since $\Theta_{\mu(f')}^0 = \varnothing$, it follows that $\mu(f')$ cannot weakly accommodate f' on its own. Therefore, the threshold of f' at $\mu(f')$ is 0 in every round of the TMDAC algorithm.

We conclude that $\mu(f')$ permanently rejects any proposal from f', which contradicts the assumption that f' is matched to $\mu(f')$ at the end of the TMDAC algorithm.

Proof of (ii) Suppose that $\ell \in \Gamma_f^j$. We need to show that $\mu(f) \succeq_f \ell$. Then, by Lemma 2(iii), $\ell \in \Gamma_f^j$ implies that one of the following two cases holds: either $\ell \in \Gamma_f^N$ or $f \in \Theta_{\ell'}^{j-1}$ for some $\ell' \in L$ such that $\ell' \succ_f \ell$.

Case 1: $\ell \in \Gamma_f^N$. In this case, by construction (Step N(c) of the Clinching Round), ℓ can weakly accommodate f alongside $\widehat{F}_\ell^f(\widetilde{\triangleright}^N) = \widehat{F}_\ell^f(\widetilde{\triangleright})$. Therefore, in every round of the TMDAC algorithm, f's threshold at ℓ is ∞ (see Algorithm 5). It follows that ℓ tentatively accepts any proposal from f in the TMDAC algorithm, hence $\mu(f) \succeq_f \ell$, as required.

Case 2: $f \in \Theta_{\ell'}^{j-1}$ for some $\ell' \in L$ such that $\ell' \succ_f \ell$. In this case, Lemma 2(iv), we have that $f \in \Theta_{\ell'}^N$ for some $\ell' \in L$ such that $\ell' \succ_f \ell$. Then, by construction (Step N(c) of the Clinching Round), ℓ' proposes to f in Step N of the Clinching Round, i.e., $\ell' \in \Gamma_f^N$. Then, f's threshold at ℓ' is ∞ in every round of the TMDAC algorithm, hence $\mu(f) \succeq_f \ell' \succ_f \ell$, as required.

Proof of (iii) Suppose that $f \in \Theta_\ell^j$. Then, by construction (Step j(c) of the Clinching Round) we have that $\ell \in \Gamma_f^j$ and, for all $\ell' \in L$ such that $\ell' \succ_f \ell$, it is the case that $\ell' \in \Delta_f^j$. From parts (i) and (ii) of the lemma, we obtain that $\mu(f) \succeq_f \ell$ and that $\mu(f) \neq \ell'$ for all $\ell' \in L$ such that $\ell' \succ_f \ell$. We conclude that $\mu(f) = \ell$, as required. □

Proof of Lemma 4

We prove the first part of the lemma by a single induction argument. For the initial step, we have that $\Theta_\ell^0 = \overline{\Theta}_\ell^0 = \varnothing$ for all $\ell \in L$.

For the induction step, let us assume that $\Theta_\ell^{j-1} \setminus \{f\} = \overline{\Theta}_\ell^{j-1} \setminus \{f\}$ holds for some $j = 1, \dots, q$ and all $\ell \in L$. We will show that $\widetilde{\triangleright}^j = \overline{\triangleright}^j$, $\widetilde{\Delta}_g^j = \overline{\Delta}_g^j$, $\widetilde{\Gamma}_g^j = \overline{\Gamma}_g^j$, and, finally, that $\Theta_\ell^j \setminus \{f\} = \overline{\Theta}_\ell^j \setminus \{f\}$ for all $g \in F$ and $\ell \in L$.

Since $j \leq q$, we have that $j - 1 < \min\{m, \overline{m}\}$. Therefore, f does not clinch any locality in Step $j-1$ of the Clinching Round with either report; therefore, $\Theta_\ell^{j-1} \setminus \{f\} = \overline{\Theta}_\ell^{j-1} \setminus \{f\}$ for all $\ell \in L$ implies that $\Theta_\ell^{j-1} = \overline{\Theta}_\ell^{j-1}$ for all $\ell \in L$. In Step $j-1$(e) of the Clinching Round the construction of $\widetilde{\triangleright}^j$ only depends on which families have clinched which localities, hence $\Theta_\ell^{j-1} = \overline{\Theta}_\ell^{j-1}$ for all $\ell \in L$ implies that $\widetilde{\triangleright}^j = \overline{\triangleright}^j$. In Step j(a), the fact that $\Theta_\ell^{j-1} = \overline{\Theta}_\ell^{j-1}$ for all $\ell \in L$ and $\widetilde{\triangleright}^j = \overline{\triangleright}^j$ implies that every locality rejects the same families under both reports so $\Delta_g^j = \overline{\Delta}_g^j$ for all $g \in F$. Similarly, in Step j(b), $\widetilde{\triangleright}^j = \overline{\triangleright}^j$ implies that $\Gamma_g^j = \overline{\Gamma}_g^j$ for all $g \in F$. Finally, consider any family $h \neq f$ in Step j(c). As $\Delta_h^j = \overline{\Delta}_h^j$, $\Gamma_h^j = \overline{\Gamma}_h^j$, and h does not misreport its preferences (only f's report changes from $\succ_f$ to $\succ'_f$), h clinches the same locality (if any), whether f reports $\succ_f$ or $\succ'_f$. We therefore conclude that $\Theta_\ell^j \setminus \{f\} = \overline{\Theta}_\ell^j \setminus \{f\}$ for all $\ell \in L$, as required.

We now turn to the second part of the lemma. Consider any $j = 1, \dots, q$ with $j < \{m, \overline{m}\}$. We have established that $\Theta_\ell^j \setminus \{f\} = \overline{\Theta}_\ell^j \setminus \{f\}$ for all $\ell \in L$. As $j < \{m, \overline{m}\}$, whether f

reports $\succ_f$ or $\succ'_f$, f does not clinch any locality in Step j of the Clinching Round. Therefore $\Theta^j_\ell = \overline{\Theta}^j_\ell$ for all $\ell \in L$, as required. □

B Additional examples

B.1 Non-existence of stable outcomes

We reproduce an example from McDermid and Manlove (2010). There are three families, two localities, and one service with the following needs and capacities:

$$\nu = {}_{s_1} \overset{\quad f_1 \;\; f_2 \;\; f_3}{\begin{pmatrix} 1 & 1 & 2 \end{pmatrix}} \quad \text{and} \quad \kappa = {}_{s_1} \overset{\quad \ell_1 \;\; \ell_2}{\begin{pmatrix} 2 & 1 \end{pmatrix}}.$$

The preferences and priorities are:

$$\succ_{f_1}: \ell_2, \ell_1, \emptyset \quad \succ_{f_2}: \ell_1, \ell_2, \emptyset \quad \succ_{f_3}: \ell_1, \emptyset, \ell_2 \quad \rhd_{\ell_1} : f_1, f_3, f_2 \quad \rhd_{\ell_2} f_2, f_1, f_3.$$

Suppose, towards a contradiction, that there exists a stable matching μ in this example. Since ℓ_2 cannot accommodate f_3, either $\mu(f_3) = \ell_1$ or $\mu(f_3) = \emptyset$. If $\mu(f_3) = \ell_1$, then $\mu(f_2) = \ell_2$ as otherwise f_2 and ℓ_2 form a blocking pair. Then, $\mu(f_1) = \emptyset$ so f_1 and ℓ_1 form a blocking pair and μ is not stable. If $\mu(f_3) = \emptyset$, then $\mu(f_1) = \ell_1$ as otherwise f_1 and ℓ_1 form a blocking pair. In turn, f_2 and ℓ_1 form a blocking pair unless $\mu(f_2) = \ell_1$. However, this matching is not stable since f_2 and ℓ_2 form a blocking pair.

One way to understand this negative result is that the choice function induced by ℓ_1's priorities does not satisfy substitutability (Roth, 1984b) because of a complementarity between f_1 and f_2. If only f_2 and f_3 compete for ℓ_1, stability dictates that f_3 be accepted and f_2 rejected. If f_1 is also competing, stability dictates that f_1 and f_2 be accepted and f_3 rejected. Thus, there is a complementarity between f_1 and f_2 in the sense that f_2 is accepted when f_1 is also considered but not otherwise.

B.2 Multidimensional Top Trading Cycles with Endowment

We illustrate the MTTCE algorithm using the example from the proof of Theorem 1. We add the following lexicographic priorities:

$$\rhd_{\ell_1} : f_1, f_2, \ldots \quad \rhd_{\ell_2} : \boldsymbol{f_1}, \boldsymbol{f_2}, \ldots \quad \rhd_{\ell_3} : \boldsymbol{f_3}, \ldots \quad \rhd_{\ell_4} : \boldsymbol{f_4}, \ldots$$

The MTTCE algorithm also depends on the order in which families are picked, should the algorithm enter the Rejection Stage. With four families, there are $4! = 24$ such orderings; however, we will see that the only part of that ordering that matters for the outcome is whether f_3 or f_4 is picked first. Therefore, we describe the MTTCE algorithm under two different orderings: MTTCE3 picks f_3 before f_4 and MTTCE4 picks f_4 before f_3.

MTTCE3

The workings of MTTCE3 are displayed in Figure 6. In Round 1, the unique trading cycle is $f_1 \to \ell_3 \to f_3 \to \ell_2 \to f_1$. The unique trading cycle is not feasible since ℓ_2 cannot accommodate f_3 alongside f_2. Therefore, the algorithm enters the Rejection Stage. As f_1

and f_2 require only one unit of capacity, they can replace any family at any locality; hence no permanent rejection occurs if one of these families is picked. In contrast, neither f_3 nor f_4 can replace f_1 at ℓ_2. By assumption, f_3 is picked before f_4 and permanently rejected by ℓ_2.

In Round 2, f_3 points at its second preference ℓ_3 and the feasible (and trivial) cycle $f_3 \to \ell_3 \to f_3$ appears so f_3 is permanently matched to ℓ_3. As a result, ℓ_3 is "full" and permanently rejects all other families, including f_1. In Round 3, f_1 points at its second preference ℓ_1 and is permanently matched to it, effectively taking advantage of ℓ_1's unassigned unit of capacity. In Round 4, since f_1 has been permanently matched, ℓ_2 points at its second-priority family f_2. In addition, $\mu^4(\ell_2) = \varnothing$ since f_1 has left ℓ_2 to be permanently matched to ℓ_1. As a result, the cycle $f_2 \to \ell_4 \to f_4 \to \ell_2 \to f_2$ is feasible so both families are permanently matched to the locality at which they are pointing. Since all families are permanently matched, the algorithm ends and produces the following matching:

$$\begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ \ell_1 & \ell_4 & \ell_3 & \ell_2 \end{pmatrix}.$$

MTTCE4

The workings of MTTCE4 are displayed in Figure 7. As for MTTCE3, the unique cycle in Round 1 is $f_1 \to \ell_3 \to f_3 \to \ell_2 \to f_1$, which is not feasible. The difference in MTTCE4 compared to MTTCE3 is that in the Rejection Stage f_4 is picked and permanently rejected by ℓ_2. In Round 2, f_4 points at its second preference ℓ_4 (which points back at f_4) so f_4 is permanently matched to ℓ_4. In Round 3, the unique cycle is again $f_1 \to \ell_3 \to f_3 \to \ell_2 \to f_1$, which is still infeasible. This time, ℓ_2 permanently rejects f_3 (because f_3 must be picked in the Rejection Stage). In Round 4, f_3 points at and is permanently matched to ℓ_3. In Round 5, ℓ_3 is "full" and permanently rejects f_1, which points at and is permanently matched to ℓ_1. In Round 6 (not displayed), f_2 and ℓ_2 points at one another and are permanently matched. The algorithm ends and produces the following matching:

$$\begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ \ell_1 & \ell_2 & \ell_3 & \ell_4 \end{pmatrix}.$$

Discussion

Observe first that both matchings Pareto dominate the endowment. This is not surprising since there is only one service and the priorities are lexicographic, hence Theorem 4 applies. Perhaps more surprising is the fact that MTTCE3 produces a chain-efficient matching, which may appear at odds with Theorem 1. What is more, we showed in the proof of Theorem 1 that no individually rational, chain-efficient, and strategy-proof mechanism exists in this specific market. However, the fact that MTTCE3 produces a chain-efficient matching in this instance does not mean it is a chain-efficient mechanism. In fact, one can show that, if f_1 reports its preferences to be $\ell_3, \boldsymbol{\ell_2}, \ldots$, MTTCE3 produces

$$\begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ \ell_2 & \ell_1 & \ell_3 & \ell_4 \end{pmatrix},$$

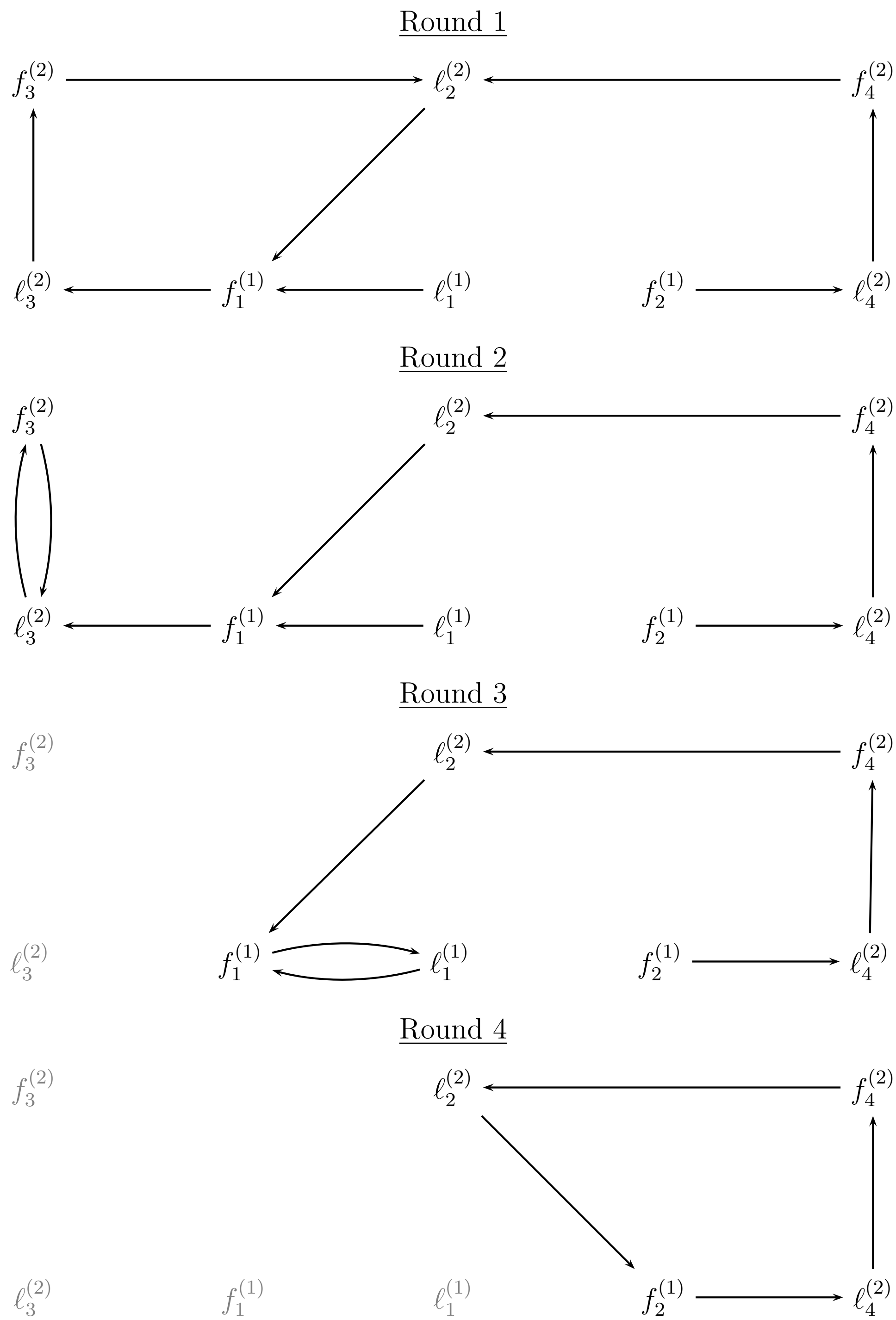

Figure 6: Workings of MTTCE3.

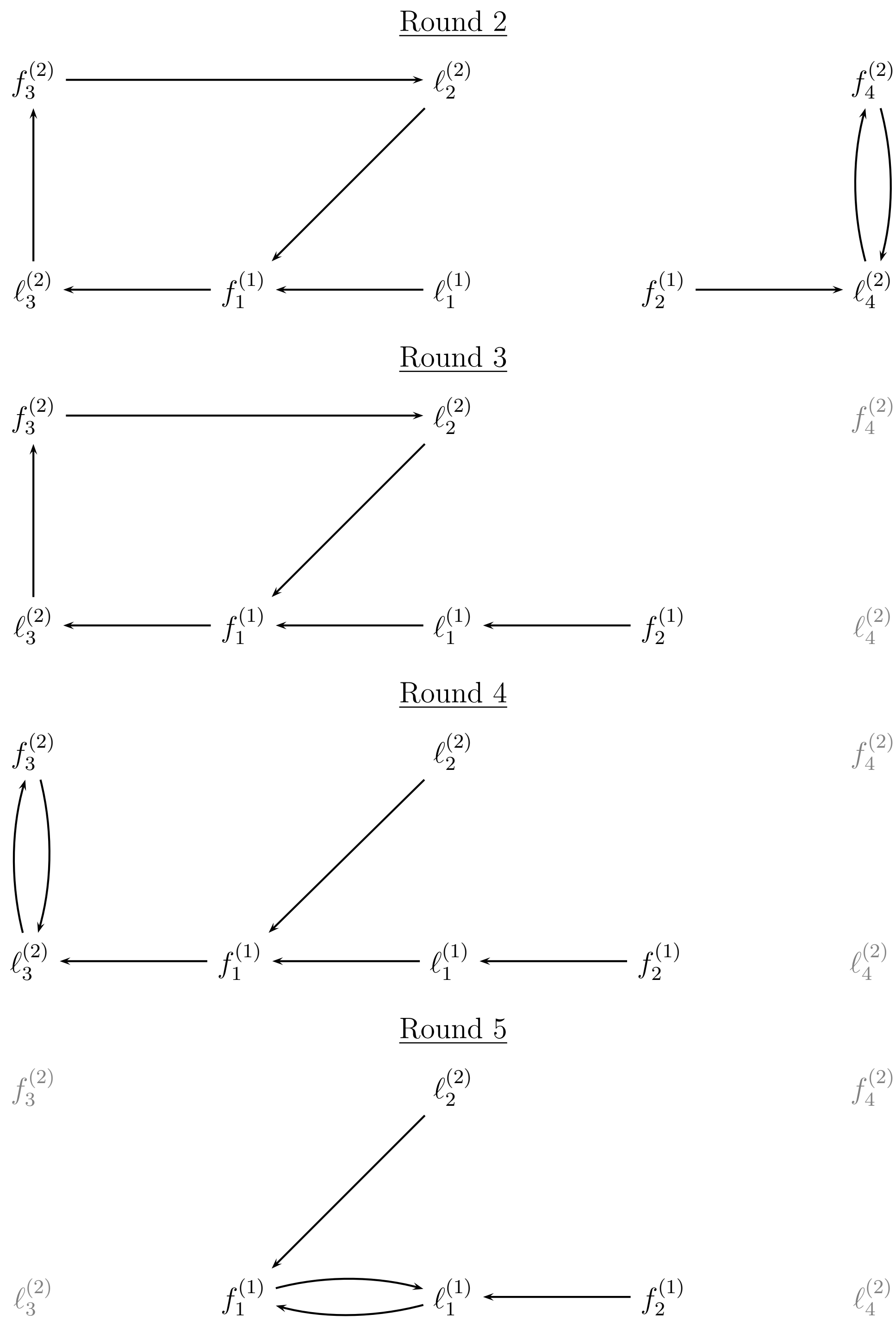

Figure 7: Workings of MTTCE4.

which is not chain-efficient.

Second, the matching produced by MTTCE3 Pareto dominates the one produced by MTTCE4. This is due to the fact that picking f_3 allows the algorithm to match f_1 to ℓ_1 in Round 3, which makes the cycle $f_2 \to \ell_4 \to f_4 \to \ell_2 \to f_2$ feasible in Round 4. In contrast, picking f_4 in MTTCE4 does not allow matching f_2 to ℓ_1 in Round 3 (since ℓ_1 points at f_1), and therefore the cycle $f_1 \to \ell_3 \to f_3 \to \ell_2 \to f_1$ remains infeasible. One might therefore wonder whether the picking order can be designed in a way that maximizes the efficiency of the mechanism. This would require picking the "best family" (from an efficiency point of view) every time the algorithm enters the Rejection Stage. Unfortunately, what constitutes the best family depends on preferences; therefore such a mechanism would violate strategy-proofness. In order for the mechanism to be strategy-proof, the picking order must be entirely independent of preferences, which has an efficiency cost. Third, the MTTCE mechanism may produce different outcomes with different priorities. If the priorities are $\succ_{\ell_1}$: $f_1, f_2, \ldots$ and $\succ_{\ell_2}$: $f_2, f_1, \ldots$ (without changing the priorities of ℓ_3 and ℓ_4 so that priorities remain lexicographic), MTTCE3 and MTTCE4 produce the same matchings as they do in our example above. If the priorities are either $\succ_{\ell_1}$: $f_2, f_1, \ldots$ and $\succ_{\ell_2}$: $f_2, f_1, \ldots$ or $\succ_{\ell_1}$: $f_2, f_1, \ldots$ and $\succ_{\ell_2}$: $f_1, f_2, \ldots$, MTTCE3 and MTTCE4 respectively produce the following matchings:

$$\begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ \ell_2 & \ell_1 & \ell_3 & \ell_4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ \ell_3 & \ell_1 & \ell_2 & \ell_4 \end{pmatrix}.$$

B.3 Threshold Multidimensional Deferred Acceptance

We illustrate the TMDA algorithm with the following example. There are seven families, four localities, and two services.

Preferences:

$$\begin{array}{llll} \succ_{f_1}: \ell_2, \ell_3, \ldots & \succ_{f_2}: \ell_4, \ell_1, \ldots & \succ_{f_3}: \ell_2, \ell_1, \ldots & \succ_{f_4}: \ell_1, \ldots \\ \succ_{f_5}: \ell_1, \ell_2, \ldots & \succ_{f_6}: \ell_1, \ell_2, \ell_3, \ldots & \succ_{f_7}: \ell_4, \ldots & \end{array}$$

Priorities:

$$\begin{array}{ll} \rhd_{\ell_1} : f_1, f_2, f_3, f_4, f_5, f_6, f_7 & \rhd_{\ell_2} : f_5, f_1, f_2, f_7, f_6, f_3, f_4 \\ \rhd_{\ell_3} : f_4, f_6, f_1, \ldots & \rhd_{\ell_4} : f_7, f_2, \ldots \end{array}$$

Services needs and capacities:

$$\nu = \begin{array}{c} \\ s_1 \\ s_2 \end{array} \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 \\ 2 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad \kappa = \begin{array}{c} \\ s_1 \\ s_2 \end{array} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ 4 & 4 & 3 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}.$$

We show that, in this example, the TMDA algorithm produces the following matching:

$$\mu^{TMDA} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 \\ \ell_3 & \ell_1 & \ell_1 & \ell_1 & \ell_2 & \ell_3 & \ell_4 \end{pmatrix}.$$

The TMDA algorithm lasts four rounds. The first three rounds are displayed in Table 5. In Round 1, every family proposes to its first-preference locality. Families f_4, f_5, and f_6

<u>TMDA - Round 1</u>

ℓ_1	$(4,2)$	θ
f_1	$(2,1)$	∞
f_2	$(1,0)$	∞
f_3	$(1,0)$	∞
$\boxed{f_4}$	$(2,0)$	2
$\boxed{f_5}$	$(1,1)$	2
$\boxed{\cancel{f_6}}$	$(1,0)$	2
f_7	$(1,0)$	0

ℓ_2	$(4,1)$	θ
f_5	$(1,1)$	∞
$\boxed{f_1}$	$(2,1)$	1
f_2	$(1,0)$	∞
f_7	$(1,0)$	1
f_6	$(1,0)$	1
$\boxed{\cancel{f_3}}$	$(1,0)$	1
f_4	$(2,0)$	0

ℓ_3	$(3,1)$	θ
f_4	$(2,0)$	∞
f_6	$(1,0)$	∞
f_1	$(2,1)$	1
$\vdots$		

ℓ_4	$(1,0)$	θ
$\boxed{f_7}$	$(1,0)$	∞
$\boxed{\cancel{f_2}}$	$(1,0)$	0
$\vdots$		

<u>TMDA - Round 2</u>

ℓ_1	$(4,2)$	θ
f_1	$(2,1)$	∞
$\boxed{f_2}$	$(1,0)$	∞
$\boxed{f_3}$	$(1,0)$	∞
$\boxed{f_4}$	$(2,0)$	3
$\boxed{\cancel{f_5}}$	$(1,1)$	0
f_6	$(1,0)$	0
f_7	$(1,0)$	0

ℓ_2	$(4,1)$	θ
f_5	$(1,1)$	∞
$\boxed{f_1}$	$(2,1)$	1
f_2	$(1,0)$	∞
f_7	$(1,0)$	1
$\boxed{\cancel{f_6}}$	$(1,0)$	1
f_3	$(1,0)$	1
f_4	$(2,0)$	0

ℓ_3	$(3,1)$	θ
f_4	$(2,0)$	∞
f_6	$(1,0)$	∞
f_1	$(2,1)$	1
$\vdots$		

ℓ_4	$(1,0)$	θ
$\boxed{f_7}$	$(1,0)$	∞
f_2	$(1,0)$	0
$\vdots$		

<u>TMDA - Round 3</u>

ℓ_1	$(4,2)$	θ
f_1	$(2,1)$	∞
$\boxed{f_2}$	$(1,0)$	∞
$\boxed{f_3}$	$(1,0)$	∞
$\boxed{f_4}$	$(2,0)$	3
f_5	$(1,1)$	0
f_6	$(1,0)$	0
f_7	$(1,0)$	0

ℓ_2	$(4,1)$	θ
$\boxed{f_5}$	$(1,1)$	∞
$\boxed{\cancel{f_1}}$	$(2,1)$	0
f_2	$(1,0)$	∞
f_7	$(1,0)$	0
f_6	$(1,0)$	0
f_3	$(1,0)$	0
f_4	$(2,0)$	0

ℓ_3	$(3,1)$	θ
f_4	$(2,0)$	∞
$\boxed{f_6}$	$(1,0)$	∞
f_1	$(2,1)$	2
$\vdots$		

ℓ_4	$(1,0)$	θ
$\boxed{f_7}$	$(1,0)$	∞
f_2	$(1,0)$	0
$\vdots$		

Table 5: Rounds 1-3 of the TMDA algorithm. Needs and capacities in parentheses. $\boxed{f_i}$: f_i proposes and is tentatively accepted. $\boxed{\cancel{f_i}}$: f_i proposes and is permanently rejected.

propose to ℓ_1. The threshold at ℓ_1 of f_1, f_2, and f_3 is ∞ since ℓ_1 can accommodate all three families together. Family f_4's threshold at ℓ_1 is 2 because ℓ_1 can weakly accommodate f_4 alongside any one of f_1, f_2, or f_3, but not alongside $\{f_1, f_2\}$ or $\{f_1, f_3\}$. Family f_5's threshold at ℓ_1 is also 2 since ℓ_1 can weakly accommodate f_5 alongside f_4 but not alongside $\{f_1, f_4\}$. Family f_6's temporary threshold at ℓ_1 is 3 since ℓ_1 can weakly accommodate f_6 alongside $\{f_4, f_5\}$ but not alongside $\{f_1, f_4, f_5\}$. However, f_6's threshold at ℓ_1 is 2 since f_6's threshold at ℓ_1 cannot exceed f_5's because f_5 has a higher priority and f_6's temporary threshold is finite. Finally, f_7's threshold at ℓ_1 is 0 since ℓ_1 cannot weakly accommodate f_7 alongside $\{f_4, f_5, f_6\}$. All three proposing families—f_4, f_5, and f_6—have a threshold of 2 at ℓ_1; therefore, ℓ_1 tentatively accepts the two higher-priority proposing families—f_4 and f_5—and permanently rejects the lowest-priority proposing family—f_6. As ℓ_1 is able to accommodate $\{f_4, f_5, f_6\}$, one might be tempted to allow ℓ_1 to tentatively accept f_6's proposal (as it would in the CMDA algorithm). However, in this case, a choice function consistent with weak envy-freeness violates cardinal monotonicity. Suppose that f_1, f_2, f_4, f_5, and f_6 propose to ℓ_1. Locality ℓ_1 cannot weakly accommodate f_4 alongside $\{f_1, f_2\}$, f_5 alongside $\{f_1, f_2, f_4\}$, and f_6 alongside $\{f_1, f_2, f_4, f_5\}$; therefore, weak envy-freeness dictates that ℓ_1 must only tentatively accept two families: f_1 and f_2. However, cardinal monotonicity dictates that at most two families can be tentatively accepted when f_4, f_5, and f_6 propose.

Families f_1 and f_3 propose to ℓ_2. Family f_5's threshold at ℓ_2 is ∞ since f_5 has the highest priority and ℓ_2 can accommodate f_5 on its own. However, ℓ_2 cannot weakly accommodate f_1 alongside f_5 as this would require two units of s_2 and ℓ_2 has only one unit available; therefore, f_1's threshold at ℓ_2 is 1. In contrast, f_2's threshold at ℓ_2 is ∞ because ℓ_2 can weakly accommodate f_2 alongside $\{f_1, f_5\}$. (Recall from Algorithm 5 that if a family's threshold is ∞, then it is allowed to exceed the thresholds of higher-priority families.) This situation illustrates how weak envy-freeness can improve efficiency over envy-freeness. Locality ℓ_2 cannot accommodate f_2 alongside $\{f_1, f_5\}$ as this would violate ℓ_2's capacity for s_2; however, as f_2 does not require any unit of s_2, ℓ_2 can weakly accommodate f_2 alongside $\{f_1, f_5\}$. The temporary threshold of f_7 at ℓ_2 is 3 since ℓ_2 can weakly accommodate f_7 alongside either one of $\{f_1, f_5\}$ or $\{f_1, f_2\}$, but not alongside $\{f_1, f_2, f_5\}$. However, since f_1's threshold is 1, we set f_7's threshold to 1 as well. The same reasoning applies to f_6 and f_3 while f_4's threshold at ℓ_2 is 0 because ℓ_2 cannot weakly accommodate f_4 alongside $\{f_1, f_3\}$. Locality ℓ_2 tentatively accepts f_1 but permanently rejects f_3.

No family proposes to ℓ_3. The threshold of both f_4 and f_6 at ℓ_3 is ∞ since ℓ_3 can accommodate $\{f_4, f_6\}$. Family f_1's threshold at ℓ_3 is 1 since ℓ_3 cannot weakly accommodate f_1 alongside f_4. Finally, f_2 and f_7 propose to ℓ_4. As f_7 has the highest priority for ℓ_4 and ℓ_4 can accommodate f_7 on its own, f_7's threshold at ℓ_4 is ∞, which means that ℓ_4 tentatively accepts f_7. In contrast, ℓ_4 cannot weakly accommodate f_2 alongside f_7; therefore f_2's threshold at ℓ_4 is 0 and ℓ_4 permanently rejects f_2.

In Round 2, f_2 and f_3 both propose to ℓ_1 after having been permanently rejected by their respective first preferences (ℓ_4 and ℓ_2) in Round 1. As a result, f_4's threshold at ℓ_1 rises to 3. This situation illustrates how a family's threshold can increase from one round to the next. In Round 1, f_4's threshold at ℓ_1 is 2 because ℓ_1 cannot weakly accommodate f_4 alongside either one of $\{f_1, f_2\}$ or $\{f_1, f_3\}$. However, in Round 2, f_2 and f_3 propose to ℓ_1 but f_1 does not. As ℓ_1 can weakly accommodate f_4 alongside $\{f_2, f_3\}$, f_4's threshold at ℓ_1 is 3. It follows that ℓ_1 continues to tentatively accept f_4. In contrast, ℓ_4 cannot weakly accommodate f_5 alongside $\{f_2, f_3, f_4\}$; therefore, f_5's threshold at ℓ_1 is 0 and ℓ_1 permanently rejects f_5. The third family that was permanently rejected in Round 1, f_6, proposes to ℓ_2 in Round 2. As f_6's threshold at ℓ_2 remains 1 and f_6 has the second-highest priority (after f_1) among proposing families, ℓ_2 permanently rejects f_6.

In Round 3, f_5 proposes to ℓ_2 and is tentatively accepted since f_5's threshold at ℓ_2 is ∞. As a result, however, ℓ_2 permanently rejects f_1 which now has a threshold of 0. Family f_6 proposes to ℓ_3 and is also tentatively accepted since f_6's threshold at ℓ_3 is ∞. A consequence of f_6's proposal to ℓ_3 is that f_1's threshold at ℓ_3 rises to 2 because ℓ_3 can weakly accommodate f_1 alongside f_6. Therefore, in Round 4, ℓ_3 tentatively accepts f_1's proposal and the algorithm ends.

B.4 Threshold Multidimensional Deferred Acceptance with Clinches

We use the example introduced in Online Appendix B.3 to illustrate the TMDAC algorithm. We show that

$$\mu^{TMDAC} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 \\ \ell_3 & \ell_1 & \ell_1 & \ell_1 & \ell_2 & \ell_2 & \ell_4 \end{pmatrix}.$$

The only difference between μ^{TMDA} and μ^{TMDAC} is that $\mu^{TMDA}(f_6) = \ell_3$ while $\mu^{TMDAC}(f_6) = \ell_2$. Since $\ell_2 \succ_{f_6} \ell_3$, $\mu^{TMDAC} \succ \mu^{TMDA}$.

The three steps of the Clinching Round are displayed in Table 6. In Step 1(a), localities ℓ_1, ℓ_2 and ℓ_3 do not reject any family since they can accommodate every family on its own. Locality ℓ_4 rejects every family that requires either two units of s_1 or one unit of s_2, i.e., ℓ_4 rejects f_1, f_4, and f_5. In Step 1(b), ℓ_1 is able to accommodate its three highest-priority families, which all receive a proposal. Locality ℓ_1 cannot weakly accommodate any other family alongside $\{f_1, f_2, f_3\}$ since all families require at least one unit of the first service; therefore none of the other families receive a proposal from ℓ_1. Locality ℓ_2 can accommodate f_5 on its own but cannot weakly accommodate f_1 alongside f_5 since both families require a unit of s_2 and only one unit is available. In contrast, ℓ_2 can weakly accommodate f_2 alongside $\{f_1, f_5\}$ because f_2 does not require any units of s_2. Therefore, ℓ_2 proposes to both f_5 and f_2. None of the other families receive a proposal as this would violate the capacity of s_1. Locality ℓ_3 is able to accommodate its two highest-priority families while ℓ_4 is only able to accommodate its highest-priority family. No other family receives a proposal from either ℓ_3 or ℓ_4 as this would violate their respective capacities for both services. Family f_7 is the only family to receive a proposal from its first-preference locality, ℓ_4. As a result, f_7 clinches ℓ_4 in Step 1(c). In Step 1(e), $\widetilde{\triangleright}^2$ is constructed by giving f_7 the lowest priority for ℓ_1, ℓ_2, ℓ_3. In particular, ℓ_2's priority list is updated to

$$\widetilde{\triangleright}^2_{\ell_2} : f_5, f_1, f_2, f_6, f_3, f_4, f_7.$$

In Step 2(a), ℓ_4 rejects all families except f_7 because f_7 has clinched ℓ_4 and ℓ_4 cannot weakly accommodate any family alongside f_7. In particular, ℓ_4 rejects its second-priority family f_2. In Step 2(b), the same proposals occur as in Step 1; however, the fact that ℓ_4 has rejected f_2 means that ℓ_1 is now f_2's most preferred locality that has not rejected f_2. As ℓ_1, proposes to f_2, f_2 clinches ℓ_1 in Step 2(c). In Step 2(e), $\widetilde{\triangleright}^3$ is constructed by moving f_2 to

Clinching Round - Step 1

ℓ_1	$(4,2)$		ℓ_2	$(4,1)$		ℓ_3	$(3,1)$		ℓ_4	$(1,0)$	
f_1	$(2,1)$	✓	f_5	$(1,1)$	✓	f_4	$(2,0)$	✓	$\boxed{f_7}$	$(1,0)$	✓
f_2	$(1,0)$	✓	f_1	$(2,1)$		f_6	$(1,0)$	✓	f_2	$(1,0)$	
f_3	$(1,0)$	✓	f_2	$(1,0)$	✓	f_1	$(2,1)$		$\vdots$		
f_4	$(2,0)$		f_7	$(1,0)$		$\vdots$					
f_5	$(1,1)$		f_6	$(1,0)$							
f_6	$(1,0)$		f_3	$(1,0)$							
f_7	$(1,0)$		f_4	$(2,0)$							

Clinching Round - Step 2

ℓ_1	$(4,2)$		ℓ_2	$(4,1)$		ℓ_3	$(3,1)$		ℓ_4	$(1,0)$	
f_1	$(2,1)$	✓	f_5	$(1,1)$	✓	f_4	$(2,0)$	✓	$\boxed{f_7}$	$(1,0)$	✓
$\boxed{f_2}$	$(1,0)$	✓	f_1	$(2,1)$		f_6	$(1,0)$	✓	f_2	$(1,0)$	✗
f_3	$(1,0)$	✓	f_2	$(1,0)$	✓	f_1	$(2,1)$		$\vdots$		
f_4	$(2,0)$		f_6	$(1,0)$		$\vdots$					
f_5	$(1,1)$		f_3	$(1,0)$							
f_6	$(1,0)$		f_4	$(2,0)$							
f_7	$(1,0)$		f_7	$(1,0)$							

Clinching Round - Step 3

ℓ_1	$(4,2)$		ℓ_2	$(4,1)$		ℓ_3	$(3,1)$		ℓ_4	$(1,0)$	
f_1	$(2,1)$	✓	f_5	$(1,1)$	✓	f_4	$(2,0)$	✓	$\boxed{f_7}$	$(1,0)$	✓
$\boxed{f_2}$	$(1,0)$	✓	f_1	$(2,1)$		f_6	$(1,0)$	✓	f_2	$(1,0)$	✗
f_3	$(1,0)$	✓	f_6	$(1,0)$	✓	f_1	$(2,1)$		$\vdots$		
f_4	$(2,0)$		f_3	$(1,0)$		$\vdots$					
f_5	$(1,1)$		f_4	$(2,0)$							
f_6	$(1,0)$		f_2	$(1,0)$							
f_7	$(1,0)$		f_7	$(1,0)$							

Table 6: Clinching Round of the TMDAC algorithm. Needs and capacities in parentheses. ✓: a locality proposes to a family. ✗: a locality rejects a family. $\boxed{f_i}$: family f_i clinches a locality.

the bottom of ℓ_2 and ℓ_3's priorities. f_2's priority list is updated to

$$\widetilde{\triangleright}^3_{\ell_2} : f_5, f_1, f_6, f_3, f_4, f_2, f_7.$$

In Step 3(a), there are no new rejections as ℓ_1 can weakly accommodate any family alongside f_2. In Step 3(b), there is one new proposal: ℓ_2 proposes to f_6 as ℓ_2 can weakly accommodate f_6 alongside $\{f_1, f_5\}$. This proposal was made possible by the fact that f_2 and f_7 have both moved below f_6 in ℓ_2's priority list. However, ℓ_2 is f_6's second preference and f_6's first preference, ℓ_1, has not rejected f_6. It follows that no new clinch occurs in Step 3(c) so the Clinching Round ends in Step 3(d) and outputs $\widetilde{\triangleright} = \widetilde{\triangleright}^3$ such that

$$\begin{array}{ll} \triangleright_{\ell_1} : f_1, f_2, f_3, f_4, f_5, f_6, f_7 & \triangleright_{\ell_2} : f_5, f_1, f_6, f_3, f_4, f_2, f_7 \\ \triangleright_{\ell_3} : f_4, f_6, f_1, \ldots & \triangleright_{\ell_4} : f_7, f_2, \ldots \end{array}$$

With the priority profile $\widetilde{\triangleright}$ constructed in the Clinching Round, the TMDA algorithm lasts four rounds. The first three rounds are displayed in Table 7. Changing the priority profile from $\triangleright$ to $\widetilde{\triangleright}$ affects the TMDA algorithm in one important way. With the constructed priority profile $\widetilde{\triangleright}$, f_6 is the third highest-priority family at ℓ_2. Since ℓ_2 can accommodate f_6 alongside $\{f_1, f_5\}$, f_6's threshold at ℓ_2 is ∞. It follows that ℓ_2 tentatively accepts ℓ_6's proposal in Rounds 2-4, which is why $\mu^{TMDAC}(f_6) = \ell_2$.

When the TMDA algorithm is run with the true priority profile $\triangleright$, f_6 does not get a threshold of ∞ at ℓ_2 because f_2 and f_7 remain above f_6 on ℓ_2's priority list. The Clinching Round's contribution in this example is to identify that f_2 and f_7 will necessarily be matched to a more preferred locality, i.e., f_2 will be matched to ℓ_1 (with $\ell_1 \succ_{f_2} \ell_2$) and f_7 will be matched to ℓ_4 (with $\ell_4 \succ_{f_7} \ell_2$). Thus, the Clinching Round identifies that f_2 and f_7 cannot cause a violation of weak envy-freeness or cardinal monotonicity when f_6 is matched to ℓ_2.

C Relationships to prior models

Our model generalizes a number of existing matching models, including the following:

- School choice (Abdulkadiroğlu and Sönmez, 2003): Every student takes up a single seat at any school. Let us relabel a student as a family and a school as a locality. In our model, this corresponds to having only one service ($|S| = 1$) and any family needing exactly one unit of the service ($\nu^f_s = 1$ for all $f \in F$).

- Controlled school choice or college admissions with affirmative action and m type-specific quotas (Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu, 2005; Westkamp, 2013): Each student is one of m types and each school has a quota for each of the m types. Let us again relabel a student as a family, a school as a locality and a type as a service. In our model, this corresponds to having m services in each locality ($|S| = m$). Each family needs exactly one unit of one of the services (ν^f_s are m-dimensional unit vectors for every $f \in F$).

- School choice with majority quotas (Kojima, 2012; Hafalir et al., 2013): Each student is either a majority or a minority student. Each school has a overall cap on the number of students, which includes a cap for majority students. Let us again relabel

TMDAC - Round 1

ℓ_1	$(4,2)$	θ
f_1	$(2,1)$	∞
f_2	$(1,0)$	∞
f_3	$(1,0)$	∞
$\boxed{f_4}$	$(2,0)$	2
$\boxed{f_5}$	$(1,1)$	2
$\boxed{\cancel{f_6}}$	$(1,0)$	2
f_7	$(1,0)$	0

ℓ_2	$(4,1)$	θ
f_5	$(1,1)$	∞
$\boxed{f_1}$	$(2,1)$	1
f_6	$(1,0)$	∞
$\boxed{\cancel{f_3}}$	$(1,0)$	1
f_4	$(2,0)$	0
f_2	$(1,0)$	0
f_7	$(1,0)$	0

ℓ_3	$(3,1)$	θ
f_4	$(2,0)$	∞
f_6	$(1,0)$	∞
f_1	$(2,1)$	1
$\vdots$		

ℓ_4	$(1,0)$	θ
$\boxed{f_7}$	$(1,0)$	∞
$\boxed{\cancel{f_2}}$	$(1,0)$	0
$\vdots$		

TMDAC - Round 2

ℓ_1	$(4,2)$	θ
f_1	$(2,1)$	∞
$\boxed{f_2}$	$(1,0)$	∞
$\boxed{f_3}$	$(1,0)$	∞
$\boxed{f_4}$	$(2,0)$	3
$\boxed{\cancel{f_5}}$	$(1,1)$	0
f_6	$(1,0)$	0
f_7	$(1,0)$	0

ℓ_2	$(4,1)$	θ
f_5	$(1,1)$	∞
$\boxed{f_1}$	$(2,1)$	1
$\boxed{f_6}$	$(1,0)$	∞
f_3	$(1,0)$	1
f_4	$(2,0)$	0
f_2	$(1,0)$	0
f_7	$(1,0)$	0

ℓ_3	$(3,1)$	θ
f_4	$(2,0)$	∞
f_6	$(1,0)$	∞
f_1	$(2,1)$	1
$\vdots$		

ℓ_4	$(1,0)$	θ
$\boxed{f_7}$	$(1,0)$	∞
f_2	$(1,0)$	0
$\vdots$		

TMDAC - Round 3

ℓ_1	$(4,2)$	θ
f_1	$(2,1)$	∞
$\boxed{f_2}$	$(1,0)$	∞
$\boxed{f_3}$	$(1,0)$	∞
$\boxed{f_4}$	$(2,0)$	3
f_5	$(1,1)$	0
f_6	$(1,0)$	0
f_7	$(1,0)$	0

ℓ_2	$(4,1)$	θ
$\boxed{f_5}$	$(1,1)$	∞
$\boxed{\cancel{f_1}}$	$(2,1)$	0
$\boxed{f_6}$	$(1,0)$	∞
f_3	$(1,0)$	0
f_4	$(2,0)$	0
f_2	$(1,0)$	0
f_7	$(1,0)$	0

ℓ_3	$(3,1)$	θ
f_4	$(2,0)$	∞
f_6	$(1,0)$	∞
f_1	$(2,1)$	2
$\vdots$		

ℓ_4	$(1,0)$	θ
$\boxed{f_7}$	$(1,0)$	∞
f_2	$(1,0)$	0
$\vdots$		

Table 7: Rounds 1-3 of the TMDAC algorithm. Needs and capacities in parentheses. $\boxed{f_i}$: f_i proposes and is tentatively accepted. $\boxed{\cancel{f_i}}$: f_i proposes and is permanently rejected.

a majority/minority student as a majority/minority family and a school as a locality. Let us also relabel "any student seats" as service s_1 and "majority student seats" as service s_2 ($|S| = 2$). In our model, then the capacity of any locality for s_1 is greater than the capacity for s_2 ($\kappa^\ell_{s_1} > \kappa^\ell_{s_2}$ for all $\ell \in L$). A majority family f needs a unit of both services ($\nu^f = (1, 1)$) whereas a minority family f' only needs a unit of s_1 ($\nu^{f'} = (1, 0)$).

- Hungarian college admissions (Biró et al., 2010): Students take up a college seat as well as a faculty seat. Both colleges and faculties have their own capacities. Let us relabel a student as a family and a college as a locality. Let us also relabel "college capacity" as the capacity of the locality for service s_1 ($\kappa^\ell_{s_1}$). Let us relabel the faculties as the remaining services $S\setminus\{s_1\}$. Therefore, each family has needs $\nu^f = (1, 0, 0, \ldots, 1, \ldots, 0, 0)$ where the second "1" is the need for a unit of one service $s \in S \setminus \{s_1\}$.

- Allocation of trainee teachers to schools in Slovakia and Czechia (Cechlárová et al., 2015): Teachers are required to teach two out of three subjects and each school has a capacity for all three subjects. Let us relabel a teacher as a family, a school as a locality, and a subject as a service. In our model, this corresponds to having three services ($|S| = 3$) and any family having needs $\nu^f_s \in \{0, 1\}$ for any two different s.

- College admission with multidimensional privileges in Brazil (Aygün and Bó, 2016): Students can claim any combination of three privileges. Colleges have quotas for each privilege, but a single student can claim more than one privilege. Let us relabel a student as a family, a college as a locality, and a privilege as a service. In our model, this corresponds to having three services ($|S| = 3$) and any family having needs $\nu^f_s \in \{0, 1\}$.

- Object allocation (Nguyen et al., 2016) or course assignment (Budish, 2011): Agents (students) demand a certain number of different objects (courses) that are supplied by a single seller (a business school). Let us relabel agents (students) as families, different objects (courses) as services, and the single seller as a single locality. In our model, this corresponds to having only one locality ($|L| = 1$).

- Resident-hospital matching with sizes (McDermid and Manlove, 2010): Doctors apply to hospitals, but the doctors can take up more than one seat at a hospital, e.g., because they arrive as couples. Let us relabel doctors as families and hospitals as localities. In our model, this corresponds to having one service ($|S| = 1$) and families having a need of arbitrary size for this service.[32]

Most of the models described above use further assumptions and develop solution approaches that suit their particular contexts but differ substantially from ours. Nevertheless, as we note throughout the paper, several impossibility and complexity results established in these papers will apply immediately to our framework.

[32]This model in turn generalizes the resident-hospital matching with inseparable couples (i.e., when couples have the same preference list and prefer to be unmatched to being in different hospitals) as well as resident-hospital matching with couples which have "consistent" preferences (McDermid and Manlove, 2010, Lemma 2.1). In both cases, we set $|S| = 1$ and $\nu^f_s \in \{1, 2\}$ in our model.