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Human Capital as an Asset Class: Implications from a General Equilibrium Model

Miguel Palacios

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Human Capital and Economic Opportunity Working Group
Economic Research Center
University of Chicago
1126 E. 59th Street
Chicago IL 60637
humcap@uchicago.edu

Human Capital as an Asset Class Implications From a General Equilibrium Model*

Miguel Palacios†

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Abstract

This paper derives the value and the risk of aggregate human capital in a dynamic equilibrium production model with Duffie-Epstein preferences. In this setting the expected return of a risky asset is a function of the asset's covariance with consumption growth and a weighted average of the asset's covariance with aggregate wage growth and aggregate financial returns. A calibration of the model matching the historical ratio of wages to consumption in the United States (85% between 1950 and 2007) suggests that the weight of human capital in aggregate wealth is 87%. The results of the calibration follow from the relative size of wages and dividends in the economy and the dynamics of the ratio of wages to consumption, which are counter-cyclical. As a result, human capital is less risky than equity, implying that the risk premium of human capital is lower than that of equity.

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†Owen Graduate School of Management, Vanderbilt University. Email: miguel.palacios@vanderbilt.edu. Phone: +1(510)717-6453. Address: 401 21st Ave South, Nashville, Tennessee, 37203 (USA). http://owen.vanderbilt.edu/palacios

1 Introduction

Wages constitute more than 80% of consumption and about 60% of gross domestic product in the United States.[1] Indeed, the present value of future wages – human capital – represents the largest share of wealth in the economy and therefore must affect portfolio choice decisions and asset prices. Furthermore, wages are the main corporate expenditure. As a result, firms have a large exposure to shocks that affect the aggregate level of wages, which affects the riskiness of cash flows and investing decisions. To take into account human capital, researchers need to make assumptions about its share of aggregate wealth and expected returns, but they often do it without a solid theoretical background. This paper provides a benchmark model, rooted in a traditional dynamic stochastic equilibrium model, through which one can analyze the drivers of human capital's value and risk.

Human capital poses a challenge for researchers because even though it is the largest single asset class in the economy, we cannot observe its value or dynamics directly; we merely observe wages, human capital's dividends. Thus, we need a framework to determine human capital's value. I use a continuous-time version of the one-sector stochastic growth model to derive the endogenous dynamics of consumption, wages, dividends, and the stochastic discount factor. I then use these dynamics to determine the value and the risk characteristics of human capital and equity.

The unobservability of human capital became less of an issue, at least for understanding its impact on asset prices, after Lucas' (1978) and Breeden's (1979) seminal contributions. If human capital had any role, it would be captured entirely by its impact on consumption's conditional growth and volatility; shocks to consumption would therefore be enough to determine the stochastic discount factor, and with it any asset's price and riskiness. However, once one departs from standard constant relative risk aversion (CRRA) preferences, the stochastic discount factor is not driven entirely by contemporaneous consumption shocks, but also on changes in the expected dynamics of consumption. Epstein

[1] Using data from the National Income and Product Accounts (NIPA) tables between 1947 and 2007, the average ratio of total compensation to consumption was 85%, and the average ratio of total compensation to production was 58%.

and Zin (1989) show that, in the special case of Kreps-Proteus (1978) preferences, shocks to consumption *and* shocks to aggregate wealth drive the stochastic discount factor. Since human capital is the largest component of aggregate wealth, not observing it continues being a problem.

Previous work exploring production economies and asset prices has not asked what the value of human capital or its riskiness is. To calculate the value of human capital the relative size of its dividend–wages–must be properly calibrated to the observed relative size of wages and consumption. Thus, I calibrate the model to match the long-term ratio of wages to consumption, consumption growth and consumption volatility. With this calibration the relative magnitude of the "dividends" paid by human capital and equity, their growth and volatility, will resemble those observed historically in the United States. What I present below are the results of that calibration.

This paper shows that under a plausible set of assumptions, we should expect human capital to be less risky than stocks. While empirical evidence is consistent with this finding (dividend growth exhibits a volatility of 10%, whereas aggregate wage growth volatility is about 2.2%),[2] the empirical literature does not provide an explanation for why wages are less volatile than dividends nor why wages perform relatively better than dividends during downturns.[3] Existing theoretical studies justify this behavior for wages and dividends by pointing out that labor contracts insure workers against idiosyncratic risk and by assuming that workers have less tolerance for risk.[4] The model I present here shows that one does not need to rely on idiosyncratic labor shocks or on ad hoc assumptions about the risk aversion of investors and workers to explain why wages are less volatile than dividends.

The intuition for the result that human capital is less risky than equity is as follows. The value of aggregate wealth, defined here as the present value of all future consumption, varies over time as technology and capital shocks affect the productivity of capital and

[2]See Campbell and Cochrane (1999) who use 11.2%. Wage data comes from "total compensation", as reported in the NIPA tables, between 1947 and 2007.

[3]See Lustig, Van Nieuwerburgh and Verdelhan (2009).

[4]See Harris and Hölmstrom (1982).

labor. For example, a shock that makes capital more productive in the future increases aggregate wealth, even though the stock of capital did not change. In the absence of frictions that make adjustments to capital costly, the value of shares on firms equal the capital stock, so the change in aggregate wealth must be absorbed by the only other claim to production: human capital. At the same time, an increase in the productivity of capital results in marginal utility being relatively higher than what it will be in the future since consumption will grow faster than on average. Thus, human capital becomes a larger part of aggregate wealth when marginal utility is relatively high, making it a "safer" investment than equity.

Another way of understanding the result is to note that the ratio of wages to consumption is counter-cyclical, implying that wages are relatively high when marginal utility is also high. The result holds as long as the elasticity of intertemporal substitution is positive but the intuition is best understood using the special case of a myopic representative agent with an intertemporal elasticity of substitution of one. In this case, the agent does not adjust the flow of consumption per unit of wealth. In the presence of decreasing returns to scale for capital, times when capital is relatively abundant are also times in which expected returns to wealth are low. Thus, by not adjusting consumption, the myopic agent induces a time-varying ratio of expected wealth returns to consumption. Moreover, expected wealth returns to consumption are counter-cyclical since capital is relatively abundant in "good times". To tie this effect to the riskiness of human capital, recall that in a competitive equilibrium with a Cobb-Douglas production function aggregate wages are proportional to gross returns to capital, which in turn move with expected wealth returns. In times of relatively abundant capital, gross capital returns are relatively low, and therefore the ratio of wages to consumption is also relatively low. The same logic applies to the case in which capital is relatively scarce. In that case wealth's returns and the competitive wage rate are relatively high, but the myopic agent does not adjust the fraction of wealth he consumes. Thus, in bad times the ratio of wages to consumption increases. Overall, wages do not fall as fast as consumption in "bad times" and do not grow as fast as consumption in "good times", acting as a hedge.

The result that human capital is less risky than equity means that we should apply a lower discount rate to it. As a result, the weight of human capital in the aggregate wealth portfolio should be, on average, at least as high as the fraction of wages to consumption observed in the data. This finding suggests that a weight for human capital closer to 87% seems more appropriate than a weight of around 70%, which is common in the literature.[5] Recent empirical work reaches a similar conclusion, estimating the weight of human capital in aggregate wealth to be more than 90%.[6] The result is relevant for the portfolio choice literature because the weight and risk of human capital affect an agent's optimal portfolio.

The model also makes transparent how human capital has an effect on asset-pricing. Duffie-Epstein preferences imply that, in addition to consumption growth, capital returns and wage growth drive the diffusion of the stochastic discount factor. Thus, any asset's risk premium will depend on the covariance of its returns with capital returns (financial wealth in this model) and wage growth. This result is consistent with empirical findings, such as Jaganathan and Wang's (1996). But information about the state of the economy, for example the ratio of wages to consumption, also affects the sensitivity of the asset to capital returns and wage growth, justifying the use of wages as a conditioning variable (Lettau and Ludvigson (2001a and 2001b), Santos and Veronesi (2006), and Julliard (2007) among others). This work has had some success explaining the cross-section of asset returns, as well as future market returns. Even though previous work finds small evidence of co-movement between human capital and asset returns (Fama and Schwert (1977), Heaton and Lucas (1996), and Davis and Willen (2000)), its dual role as a factor and as a conditioning variable appears to be important for explaining asset returns.

The use of Duffie-Epstein preferences allows me to calibrate reasonable values for the risk-free rate an the equity premium, but does not drive the main result: if wages are a larger fraction of consumption when marginal utility is high, then human capital is less

[5]For example, see Baxter and Jermann (1997) and Chen et al. (2008).
[6]See Lustig, Van Niewerburgh and Verdelhan (2009).

risky than equity; if human capital pays, on average, 85% of consumption and its risk is lower than that of equity then human capital's weight in the aggregate wealth portfolio should not be, on average, less than 85%.

The structure of the rest of the article is as follows. Section 2 relates this paper to the existing literature. Section 3 presents the model with an infinitely-lived agent and derives the dynamics of the general equilibrium. Section 4 discusses the main implications from the model, and Section 5 analyzes the numerical calibration.

2 Relation to existing literature

The literature that originally tackled the impact of human capital on asset prices started from assumptions about the exogenous wage process (Mayers (1972)). Fama and Schwert (1977) tested the empirical predictions of Mayers' model and concluded that human capital, as proxied by wages, did not play a major role in determining asset prices.

More recent work, recognizing the weakness of treating wages as a proxy for human capital, tries to include human capital as relevant for determining other assets' prices. The two main characteristics of human capital that matter most for asset-pricing are its weight in the aggregate wealth portfolio and its riskiness.

The weight of human capital in the aggregate wealth portfolio is either assumed exogenously or derived endogenously under a restricted set of assumptions. Estimates of this weight typically range between 60% and 80%, all lower than the estimate presented here (Baxter and Jermann (1997), Lettau and Ludvigson (2001), Lustig and Van Nieuwerburgh (2006), Chen et al.(2008)). Two empirical papers that support higher values for the fraction of human capital in aggregate wealth are Jorgenson and Fraumeni (1989) and Lustig, Van Nieuwerburgh and Verdelhan (2010). This paper finds similar results, while highlighting the mechanism that drives them.

The riskiness of human capital is comprised of idiosyncratic and systematic components. In my model I ignore idiosyncratic shocks, so systematic risk is the sole determinant of human capital's expected returns. Empirical studies require assumptions about human capital's expected returns. Shiller (1995) assumes that human capital's expected returns are constant. Campbell (1996) assumes that its expected returns are conditioned to be the same as those of stocks, while Jagannathan and Wang (1996) assume human capital's return is equal to labor income growth. Palacios-Huerta (2003a) uses the increase in labor income attributed to an extra year of education as a proxy for human capital returns and Lustig and Van Nieuwerburgh (2006) assume it to be the return that minimizes pricing errors in their model.[7] The model presented here provides theoretical foundations to evaluate these assumptions. In particular, human capital's expected returns appear to be smaller than those of stocks, and innovations to expected returns of human capital and equity are negatively correlated.

Human capital is not only important on an aggregate level for asset-prices, but also important for portfolio choice. The portfolio choice literature assumes exogenous wage dynamics (for example, Merton (1971), Svensson (1988), Koo (1998), Campbell and Viceira (1999), and Viceira (2001)), but in that work wages and asset prices can diverge unrealistically. Benzoni, Collin-Dufresne and Goldstein (2007) provide a solution to this problem by assuming cointegration between wages and financial markets. However, the process driving the cointegration is also exogenous. In contrast, my model delivers cointegration endogenously, providing an intuitive reason for why wages and dividends follow a trend over time.

The model is grounded in the traditional one-sector stochastic growth literature.[8] This literature goes back at least to Ramsey (1928) and the stochastic versions of Mirrlees (1967), Brock and Birman (1972) and Merton (1975). Subsequent work tried to bridge macroeconomic observations with asset prices in an attempt to address the shortcomings pointed out by Mehra and Prescott (1985) on the predictions of Rubinstein (1976), Lucas

[7]Palacios-Huerta (2003b) uses the same method to compare the returns to investing in education of different demographic groups.

[8]A general closed-form solution to the growth model is not known, though solutions do exist restricting the parameters (Smith (2007)). The restrictions are not useful to obtain reasonable results, so I proceed with a numerical solution.

(1978) and Breeden's (1979) consumption-based model.[9] Importantly, none of these papers explored the implications for the value and expected return of human capital.

The paper also links to more recent work that explores the asset pricing implications of having multiple sources of income that add up to aggregate consumption, as in Cochrane, Longstaff and Santa Clara (2006), Başak (1999), Gomes et al.(2007), Gârleanu and Panageas (2008), and Santos and Veronesi (2006). In contrast to this work, I allow for capital accumulation, addressing the link between capital growth, production shocks, and human capital returns.

The implications for asset prices derived from the model follow empirical observations linking price-dividend ratios, forecasted economic growth, and asset returns (Fama and French (1989), Fama (1990), Lamont (1998), Lettau and Ludvigson (2001a and 2001b) and Santos and Veronesi (2006)). The theoretical model I present here is a step towards understanding the dynamics observed in the data.

3 Model description

This section describes and solves the equilibrium model, characterizes the representative agent's optimal consumption decision, and derives joint dynamics for wages and dividends.

3.1 Economic environment

Consider a competitive, continuous-time economy with a continuum of identical agents, whose mass is normalized to 1, and a continuum of identical firms. Each agent works and chooses to invest her wealth W_j optimally among the existing firms. Agents can trade claims that span all possible outcomes in the economy, and therefore markets are complete. We are interested in three of these claims. The first claim is "human capital," and it is defined as a security that pays aggregate wages. The second claim is "equity," and it

[9] See Brock (1982), Cochrane (1991, 1996), Belo (2007), Rouwenhorst (1995), Jermann (1998), Boldrin, Christiano and Fisher (2001), Danthine and Donaldson (1992 and 2002).

consists of a claim to aggregate dividends. The last claim is an instantaneously risk-free bond.

3.2 Production and profits

The single productive technology produces a single consumption good. At any point in time, the agent can consume the good or invest it in any of the firms. Capital investment is perfectly reversible, so changes in its stock are costless. Output at each point in time depends on the amount of aggregate capital, labor, the technology level and production shocks. Denote by $(\Omega, \Im, \mathbb{P})$ a fixed complete probability state, and the stochastic process $(B_t)_{t\geqslant 0}$, a standard 2-dimensional Brownian motion with respect to the filtration $(\mathcal{F}_t)$. The first element of the Brownian motion will be denoted by dB_A and the second by dB_K. The correlation between the shocks is $\varphi_{K,A}$. The Brownian motion reflects shocks to technology and output as described below.

I denote the technological level by A_t, which follows a geometric brownian motion:

$$dA_t = A_t \eta dt + A_t \sigma_A dB_A, \tag{1}$$

where η is the growth rate of the technology level. Under this specification, shocks to the technology level are permanent.

The dynamics of firm j's capital are given by:

$$dK_{t,j} \quad = \quad K_{t,j}(Z(K_{t,j}, L_{t,j}, A_t) - y_t L_{t,j} - \delta - d_{t,j})dt + \sigma_K K_{t,j} dB_{K,t} \tag{2}$$

where $Z(\cdot)$ is output per unit of capital, y_t is the wage rate per unit of capital, δ is the depreciation rate, $d_{t,j}$ are dividend payments per unit of capital, and σ_K is the volatility associated with the stochastic component of output.[10] Output per unit of capital has

[10]This specification results from output following the following stochastic process:

$$K_{t,j} Z(K_{t,j}, L_{t,j}, A_t)dt + K_{t,j}\sigma_K dB_{K,t}. \tag{3}$$

A firm's accumulation of capital is simply its output *minus* wages paid, depreciation, and dividends a deterministic component $Z(K_t, L_t, A_t)$, and a stochastic one $\sigma_K dB_{K,t}$. The stochastic component of output can be interpreted in several ways. Output can be lower than expected due to a transient fall in the technology level. If we interpret the weather as part of what we call "technology", then unexpectedly poor weather leads to lower than expected output. Output can also be lower than expected due to an unforseen problem with capital itself, for example, if a machine breaks down. Alternatively, the $dB_{K,t}$ shocks can be interpreted as coming from a stochastic depreciation process. If $\sigma_K = 0$, then the technology is instantaneously riskless and equity's return will equal the risk-free rate. The evolution of technology is exogenous, and its shocks capture uncertainty in the rate at which it evolves. This production specification is flexible, allowing for a stochastic component of production technological improvement.

The literature concentrates on specifications for output that are continuous, differentiable, and homogeneous of degree one in K and AL. Here, I will study the widely used labor-augmenting Cobb-Douglas specification,

$$K_t Z(K_t, L_t, A_t) = K_t^{\alpha}(A_t L_t)^{1-\alpha}. \tag{4}$$

The choice of making the technology labor-augmenting instead of Hicks-neutral or capital-augmenting does not affect the main results, but it affects the interpretation of the coefficients of A_t. The particular assumption for the production technology does impact wage and dividend dynamics, as different specifications for $Z(K_t, L_t, A_t)$ result in different shares of output being paid to labor and capital. The specification presented here should be interpreted as a benchmark case, with richer structures possible when the shares of output that go to labor and capital change over time.

I assume individuals have no disutility from working, so they do not face a work-

returned to investors. It is perhaps more natural to use output shocks proportional to the output level, or proportional to the impact that A_t has on output, instead of shocks proportional to capital. The choice presented here is a simplification that keeps the instantaneous variance of capital constant. The alternative specification leads to stochastic volatility which adds one more layer of complexity to the results and makes the numerical results less stable, without necessarily adding interesting insights.

leisure tradeoff.

3.3 Firms

Firms are run by managers who maximize the present value of the firm by maximizing the expected present value of dividends. Managers choose how much labor to hire at every point in time and how much capital to return to investors as dividends, or how much equity to raise for investment. Managers take as given the dynamics of wages and the stochastic discount factor. Following the previous discussion, denoting M_s as the stochastic discount factor, the manager's problem is:

$$\max_{\{L_j\}_t^\infty,\{d_j\}_t^\infty} E_t\left[\int_t^\infty M_s K_{s,j} d_{s,j} ds\right] \tag{5}$$

$$s.t. \qquad dK_{t,j} \;=\; K_{t,j}(Z(K_{t,j}, L_{t,j}, A_t) - y_t L_{t,j} - \delta - d_{t,j})dt + \sigma_K K_{t,j} dB_t. \tag{6}$$

3.4 The Representative Agent

I assume the agent maximizes his lifetime value of consumption with Duffie-Epstein recursive utility over an infinite horizon. Because there are only two sources of uncertainty (dB_A and dB_K), equity, human capital, and a risk-free bond will span all possible contingent claims. Without loss of generality, I assume the representative agent only trades these claims in setting up its optimization problem. The agent's problem is:

$$\begin{aligned} V_t &= \max_{\{C_s,x_s\}_t^\infty} \int_t^\infty f(C_s, V_s) ds \\ s.t. \quad dW_t &= (W_t(\mathbf{x_t}'(\mathbf{r_t} - r_{f,t}) + \mathbf{1}' r_{f,t}) - C_t)dt + W_t \mathbf{x_t}' \sigma_\mathbf{r} \mathbf{dB}, \end{aligned} \tag{7}$$

where, following Duffie and Epstein (1992a and 1992b),

$$f(C,V) = \frac{\beta}{1-\frac{1}{\psi}}(1-\gamma)V((C((1-\gamma)V)^{\frac{-1}{1-\gamma}})^{1-\frac{1}{\psi}} - 1), \tag{8}$$

W_t is an agent's wealth, $\mathbf{x_t}$ is the vector with the fraction of wealth invested in equity, human capital, and the risk-free rate, $\mathbf{r_t}$ is the vector of expected returns to equity and human capital, $\mathbf{1}$ is a vector of ones, and $\sigma_{\mathbf{r}}$ is the covariance matrix of asset return volatility associated with the dB_A and dB_K shocks.

This specification is the continuous-time equivalent of Epstein-Zin preferences (see Epstein and Zin, 1989, and Kreps and Porteus, 1978). The advantage of using Duffie-Epstein preferences is that the elasticity of intertemporal substitution and the coefficient of relative risk aversion can vary independently of each other. With CRRA preferences (which in the present setting are the special case of $\gamma = 1/\psi$) an increase in relative risk aversion necessarily implies a decrease in the elasticity of intertemporal substitution. Risk-aversion and intertemporal substitution are different concepts, and Duffie-Epstein preferences give us the flexibility to treat them as such. The extra degree of freedom provided by an independent elasticity of intertemporal substitution has been shown to improve the calibration of equilibrium models (see Banzal and Yaron, 2004), in particular a low risk-free rate and a high equity premium.

3.5 Equilibrium

This section defines and proceeds to characterize the equilibrium in this economy. In equilibrium, the supply of capital by agents equals the demand of capital by firms. This condition is equivalent to agents consuming all their income from dividends and wages (a "no free-disposal" condition). lastly, the value of capital held far away in the future converges to zero (the "transversality" condition).

3.5.1 Definition

Capital and labor markets must clear in equilibrium. The definition that follows is standard:

Definition 1 *In this economy, an equilibrium is defined as a stochastic path for*

$\{K_t, L_t, A_t, d_t, \mathbf{x_t}, \sigma_{\mathbf{r},\mathbf{t}}, \mathbf{r_t}, C_t, y_t, M_t\}_t^\infty$ *such that, for every* t,

1. *Given the processes for* $\{y_t,\ A_t,\ K_t,\ M_t\}$, *each firm chooses* L_t *and* d_t *to maximize the present value of dividends (Equation (5)).*

2. *Given the processes for* $\{K_t, \mathbf{x_t}, \sigma_{\mathbf{r},\mathbf{t}}, \mathbf{r_t}, y_t\}$, *the agent chooses* C_t *and* $\mathbf{x_t}$ *to maximize his expected lifelong utility.*

3. *Capital markets clear, which implies consumption equals wages plus dividends:* $C_t = K_t(y_t + d_t)$.

4. *Labor markets clear:* $L_t = 1$.

3.5.2 Capital Dynamics

Capital dynamics will be the result of managers optimally choosing the capital and labor required for production over time. Throughout the remaining portion of the paper it will be convenient to introduce the state variable

$$Z_t = \left(\frac{A_t L_t}{K_t}\right)^{1-\alpha}. \tag{9}$$

Given the production function, Z_t is output per unit of capital. As I show below, output per unit of capital is the only state variable in the economy. Yet, for mathematical convenience, the main results are expressed as a function of

$$z_t \equiv \frac{1}{1-\alpha} \log Z_t, \tag{10}$$

noting that output per unit of capital is $e^{(1-\alpha)z_t}$. Applying the Itô-Doeblin Lemma to z_t in equation (10), its dynamics are given by:

$$dz_t = \left(c_t - e^{(1-\alpha)z_t} + \eta + \delta + \frac{\sigma_K^2}{2}\right) dt + \sigma_A dB_{A,t} - \sigma_K dB_{K,t}, \tag{11}$$

where

$$c_t = \frac{C_t}{K_t}. \tag{12}$$

The instantaneous variance of Z_t drives many results in the paper so it is useful to define it here. Denoting Ω^2 as Z_t's variance, it equals:

$$\Omega^2 = \sigma_A^2 + \sigma_K^2 - 2\sigma_A\sigma_K\varphi_{K,A}. \tag{13}$$

When output per unit of capital, Z_t, is very small, consumption dominates any amount produced (0 when $z_t \to 0$), and the drift of z_t is positive. On the other hand, as z_t grows consumption does not grow as much, since the agent faces a higher opportunity cost of consuming. Thus, for large values of z_t, its drift is negative. As a result, z_t is mean-reverting. Since z_t is the only state variable, all the relevant quantities in the model – the risk-free rate, the equity premium, the ratio of wages to consumption, and the dividend price ratio – are mean reverting as well.

3.5.3 Optimization solution

The manager's optimization problem can be solved using the standard dynamic programming procedure as in Merton (1973). The manager's value function will depend on the firm's capital at time t, $K_{j,t}$, aggregate capital in the economy K_t, time, and technology efficiency A_t. Let $J(K_{j,t}, K_t, t, A_t)$ be the solution of the manager's Bellman equation. The following proposition characterizes the solution of the manager's and representative agent's optimization problem:[11]

Proposition 1 *Assume $\psi > 0$, $\psi \neq 1$, $\gamma > 0$ and $\gamma \neq 1$, if the value function $J(K_{j,t}, K_t, t, A_t)$*

[11] Note that wages do not enter into the Jacobian as on aggregate these depend completely on changes to technology and the capital stock (this result is derived in the appendix), and thus we only need to include A_t. If technology was irreversible, or if wages were subject to frictions, then the level of these variables would be needed here.

exists and is twice continuously differentiable, then the manager's value function can be expressed as:

- $J(K_{j,t}, K_t, t, A_t) = J(K_{j,t}, z_t)$
- $J(K_{j,t}, z_t) = K_{j,t} m(z_t)$
- $m(z_t) = g(z_t) - \frac{g'(z_t)}{1-\gamma}$
- $g(z_t)$ *solves the following ODE:*

$$\begin{aligned} 0 \; = \; & g(z)(1-\gamma)\left(\frac{\beta\psi}{1-\psi} + e^{(1-\alpha)z} - \frac{c(z)}{1-\psi} - \delta - \gamma\frac{\sigma_K^2}{2}\right) + \\ & g'(z)\left(-e^{(1-\alpha)z} + \frac{c(z)}{1-\psi} + \delta + \eta + \gamma\frac{\sigma_K^2}{2} - \gamma\frac{\sigma_A^2}{2} - (1-\gamma)\frac{\Omega^2}{2}\right) + \\ & g''(z)\frac{\Omega^2}{2} \end{aligned} \tag{14}$$

with $g(\infty) = 0$ *and* $g'(\infty) = 0$ *as boundary conditions.*

Optimal consumption will be given by

$$c_t^* \equiv \frac{C_t^*}{K_t} = \beta^{\psi} \frac{g(z_t)^{\frac{1-\gamma\psi}{1-\gamma}}}{m(z_t)^{\psi}}. \tag{15}$$

Proof: See Appendix.

The boundary conditions follow from the agent choosing to consume more when output per unit of capital increases. As output per unit of capital becomes arbitrarily large, the marginal utility approaches zero, implying that $g(z)$ approaches zero for large values of z.

Equation (15) is a generalization of the CRRA result that relates the stochastic discount factor with consumption. When $\gamma = 1/\phi$, the function $g(z_t)$ dissappears and consumption becomes only a function of constants and the stochastic discount factor. In the more general setting with Duffie-Epstein preferences $g(z_t)$ enters into the equation and introduces an additional component linking consumption and the stochastic discount factor. As a result, the risk-free rate and the market price of risk will not be characterized any more by consumption growth alone.

3.6 Dynamics of the economy

Having determined the optimal rule for choosing between consumption and saving, we now analyze the dynamics of the economy. The optimal consumption rule will determine the dynamics of consumption and the stochastic discount factor, which determines the risk-free rate. The optimal consumption rule in conjunction with the marginal product of labor determines the dynamics of wages and dividends, which in turn determine the value of equity and human capital. Once we determine the value of equity and human capital, we can study the equity premium and the co-movement between asset returns and human capital. I derive all these results in this section.

3.6.1 The stochastic discount factor, and the risk-free rate

The dynamics of the stochastic discount factor derived in Duffie and Epstein (1992a) can be expressed as:

$$\frac{dM_t}{M_t} = \frac{df_C(C,V)}{f_C(C,V)} + f_V(C,V)dt. \tag{16}$$

We find the dynamics of the stochastic discount factor applying Itô to the expression in equation (16). Before presenting the result, the following corollary is helpful in simplifying the expressions that follow.

Corollary 1 *$m(z_t)$ satisfies the following ODE:*

$$\begin{aligned} 0 \;=\; & m(z)\left(\frac{\beta(1-\gamma)\psi}{1-\psi} - \frac{1-\gamma\psi}{1-\psi}g(z)^{-1}c(z)m(z) + \alpha e^{(1-\alpha)z} - \delta - \gamma\mu_K - \frac{1}{2}\gamma(1-\gamma)\sigma_K^2\right) + \\ & m'(z)\left(\eta - \mu_K - \frac{\sigma_A^2}{2} + \frac{\sigma_K^2}{2} + (1-\gamma)(\sigma_A\sigma_K\varphi_{K,A} - \sigma_K^2)\right) + \\ & m''(z)\frac{1}{2}\Omega^2 \end{aligned} \tag{17}$$

with $m(\infty) = 0$ *and* $m'(\infty) = 0$.

Proof: See appendix.

Using the corollary and applying Itô, the dynamics of the stochastic discount factor are given by:

$$\frac{dM_t}{M_t} = \mu_{M,t}dt + \sigma_A \frac{m'(z_t)}{m(z_t)} dB_A - \sigma_K \left(\gamma + \frac{m'(z_t)}{m(z_t)} \right) dB_K, \tag{18}$$

where

$$\mu_{M,t} = -(\alpha e^{(1-\alpha)z_t} - \delta - \gamma \sigma_K^2) + \frac{m'(z_t)}{m(z_t)} (\sigma_K^2 - \sigma_K \sigma_A \varphi_{K,A}). \tag{19}$$

To make the analysis that follows clearer, we can find the dynamics of consumption and compare them to the dynamics of the stochastic discount factor. In particular, we can compare the diffusion terms of consumption and the stochastic discount factor. Using equations (14), (15), and (58), and substituting, the volatility of the diffusion term of the stochastic discount factor can be expressed as:

$$\sigma_{\mathbf{M,t}} = -\frac{1}{\psi} \begin{pmatrix} \sigma_{A,c} \\ \sigma_{K,c} \end{pmatrix} + \left(\frac{1}{\psi} - \gamma \right) \begin{pmatrix} 0 \\ \sigma_K \end{pmatrix} + \frac{(1-\gamma\psi)}{\psi(1-\gamma)} \frac{g'(z_t)}{g(z_t)} \begin{pmatrix} \sigma_A \\ -\sigma_K \end{pmatrix}, \tag{20}$$

where $\sigma_{A,c}$ and $\sigma_{K,c}$ denote the instantaneous volatility of consumption due to shocks to A_t and K_t, respectively.

Equation (20) implies that the market price of risk, given by the volatility of the stochastic discount factor, will not only be driven by the volatility of consumption as in the CRRA case, but will depend on two additional components. The first one, $\left(\frac{1}{\psi} - \gamma \right) (0, \sigma_K)'$, is volatility in the stochastic discount factor due to capital's volatility, which, as is shown later, is also the volatility of equity. Thus, the market price of risk depends on the volatility of financial wealth. The second one, $\frac{(1-\gamma\psi)}{\psi(1-\gamma)} \frac{g'(z_t)}{g(z_t)} (\sigma_A, -\sigma_K)'$, is volatility in the stochastic discount factor due to volatility in capital's productivity Z_t. Below I show that this term can be expressed as a function of shocks to aggregate wages. In the special case of CRRA preferences, $\frac{1}{\psi} = \gamma$, the last two components of volatility in the stochastic discount factor become 0, leaving us with the benchmark case of the Consumption CAPM.

The additional components in the volatility of the stochastic discount factor are closely related to results found in Duffie and Epstein (1992) and Bansal and Yaron (2004). These authors note that Duffie-Epstein preferences (Epstein-Zin in the case of Bansal and Yaron), imply that the stochastic discount factor's volatility depends on consumption's volatility and aggregate wealth's volatility. Equation (20) shows the same result, with the additional structure imposed by the production model.

The risk-free rate is the negative of the stochastic discount factor's drift. Rearranging terms, r_f will be given by:

$$r_{f,t} \;=\; \alpha e^{(1-\alpha)z_t} - \delta - \gamma\sigma_K^2 - \frac{m'(z_t)}{m(z_t)}(\sigma_K^2 - \sigma_K\sigma_A\varphi_{K,A}). \tag{21}$$

Equation (21) states that the risk-free rate is the expected return on physical capital adjusted by capital's risk premium. Unlike the CRRA case, the risk-free rate will not be a simple function of consumption growth.

3.6.2 Wages, human capital, and equity

Given the dynamics of the stochastic discount factor, we can now turn our attention to wages and the value of equity. Wages can be found using the manager's first order condition with respect to labor. In this economy, competition between firms ensures that wages equal the marginal product of labor. For the Cobb-Douglas case this implies that wages per unit of capital will be:[12]

$$y_t = (1-\alpha)e^{(1-\alpha)z_t}. \tag{22}$$

The value of a claim to aggregate human capital is the present value of the dividend

[12] This result is derived in Appendix B.

paid by this claim. Thus, the value of aggregate human capital is:

$$H_t = E_t \left[\int_t^\infty ((1-\alpha)e^{(1-\alpha)z_\tau} K_\tau) \frac{M_\tau}{M_t} d\tau \right] \tag{23}$$

The value of equity equals the present value of net dividends, which can be negative. This is the case when capital needs to be raised (i.e., when the difference between production and the sum of investment and wages is negative.

Following the previous paragraph, and using the fact that in equilibrium dividends and wages add up to consumption, net dividends per unit of capital will be:

$$d_t = c(z_t) - e^{(1-\alpha)z_t}(1-\alpha), \tag{24}$$

and equity's value is:

$$S_t = E_t \left[\int_t^\infty (K_\tau c(z_t) - K_\tau e^{(1-\alpha)z_\tau}(1-\alpha)) \frac{M_\tau}{M_t} d\tau \right] \tag{25}$$

Given this information, we can derive the dynamics of human capital and equity using the stochastic differential equations implied by their definition as the present value of wages and dividends, and by the fact that the state of the economy is captured by the level of capital and z_t. The following proposition summarizes this result:[13]

Proposition 2 *Let $H(K_t, z_t)$ denote the value of a claim to human capital, $\Sigma_1 = \sigma_A \frac{m'(z)}{m(z)}$ and $\Sigma_2 = -\sigma_K \left(\gamma + \frac{m'(z)}{m(z)}\right)$. Then the value of human capital will be characterized by*

- *$H(K_t, z_t) = K_t h(z_t)$.*
- *$h(z_t)$ is given by the solution to the following ODE:*

$$0 = (1-\alpha)e^{(1-\alpha)z} + h(z)(\mu_K(z) - r_f(z) + \sigma_K(\Sigma_2 + \Sigma_1 \varphi_{K,A})) \tag{26}$$

[13] A more frequent method is using the condition that under the risk-neutral measure, the expected return of a claim to equity *cum* dividend must equal the risk-free rate. This is, of course, equivalent to what I do above.

$$+h'(z)\left(\eta - \mu_K(z) - \frac{\Omega^2}{2} + \sigma_A(\Sigma_1 + \Sigma_2\varphi_{K,A}) - \sigma_K(\Sigma_2 + \Sigma_1\varphi_{K,A})\right)$$
$$+h''(z)\frac{\Omega^2}{2},$$

subject to: $h(-\infty) = 0$ *and* $h'(-\infty) = 0$.

- *The value of equity will be given by* $S_t = K_t$.

Proof: See Appendix.

The boundary conditions in Proposition 2 follow from wages approaching zero in equilibrium as output per unit of capital approaches zero. Dividends do not approach zero, since the agent still chooses to consume some of his capital. Therefore, a claim to equity will have value when $z \to \infty$, but a claim to human capital will not.

Given $h(z_t)$ and S_t, it is trivial to calculate the weight of human capital in the aggregate wealth portfolio. It will equal:

$$W_{hc} = \frac{h(z_t)}{1 + h(z_t)} \tag{27}$$

4 Human capital and the equity risk premium

Having derived the dynamics of the economy, in particular the stochastic discount factor and the claims of human capital and equity, I explore the implications of those dynamics for the expected returns for both claims. Several papers make assumptions about the value and expected returns to human capital with the purpose of estimating the return to the aggregate, human capital-inclusive, wealth portfolio. For that purpose Jagannathan and Wang (1996) assume that the realized human capital return is equal to wage growth while Campbell (1996) assumes that expected human capital returns are identical to expected asset returns. We can compare these assumptions to what the model predicts.

In equilibrium, expected excess returns to any claim are equal to the negative of the covariance between the stochastic discount factor and the claim's returns. We can use this result to express any claim's expected excess return as a function of the state of the economy. Using Equation (58) and Equation (20), claim S's risk premium is:

$$\begin{aligned} r_{e,t} - r_{f,t} &= \frac{1}{\psi} cov_t \left(\frac{dC}{C}, \frac{dS}{S} \right) + \left(\gamma - \frac{1}{\psi} \right) cov_t \left(\frac{dK}{K}, \frac{dS}{S} \right) + \\ & \frac{(\gamma - \frac{1}{\psi})}{(1-\gamma)} \frac{g'(z_t)}{g(z_t)} cov_t \left(dZ, \frac{dS}{S} \right). \end{aligned} \tag{28}$$

Noticing that $cov(dZ, \frac{dS}{S}) = \frac{1}{1-\alpha}(cov(\frac{dY}{Y}, \frac{dS}{S}) - cov(\frac{dK}{K}, \frac{dS}{S}))$, and that $cov_t\left(\frac{dK}{K}, \frac{dS}{S}\right) = \sigma_K^2$, the excess return of claim S can be reexpressed as:

$$\begin{aligned} r_{e,t} - r_{f,t} &= \frac{1}{\psi} cov_t \left(\frac{dC}{C}, \frac{dS}{S} \right) + \\ & \left(\gamma - \frac{1}{\psi} \right) \left(w_{y,t} cov_t \left(\frac{dY}{Y}, \frac{dS}{S} \right) + (1 - w_{y,t}) cov_t \left(\frac{dK}{K}, \frac{dS}{S} \right) \right), \end{aligned} \tag{29}$$

where $w_{y,t} = \frac{1}{(1-\gamma)(1-\alpha)} \frac{g'(z_t)}{g(z_t)}$. Equation (29) separates the risk-premium into two components. The first one corresponds to the standard CRRA result (with $\gamma = \frac{1}{\psi}$) relating the risk premium to the covariance of the value of a claim with consumption, scaled by the inverse of the intertemporal elasticity of substitution. The second term is only relevant when $\gamma \neq \frac{1}{\psi}$, that is, when the intertemporal elasticity of substitution and the coefficient of relative risk aversion are not linked through only one parameter as with CRRA preferences.

The second term can be interpreted in light of the growing long-run risk literature. Long-run risk is captured in aggregate wealth returns, which in turn are a function of financial returns and wage growth. The second term in equation (29) is the weighted average of two components, the covariance of an asset with aggregate wage growth and the covariance of an asset with financial returns.

Equation (29) predicts that financial returns and wage growth will explain the cross-section of stock returns, after controlling for consumption growth. It also shows that the relationship will be conditional, as both the weight assigned to wage growth, $w_{y,t}$, and the covariance between wage growth and asset returns, changes over time. Equation (29) predicts that empirical tests of the Consumption CAPM, and the CAPM, that use wage growth, either directly in the regression (as in Jagannathan and Wang (1996)), or as a conditioning variable (Lettau and Ludvigson, 2001b), will better explain the cross-section of asset returns, even though neither the Consumption CAPM nor the CAPM hold in this setting.

Moving forward to the calibration of the model, for the special case of a claim to aggregate dividends, equation (28) can be simplified to:

$$r_{e,r_f} = r_{e,t} - r_{f,t} \quad = \quad \gamma \sigma_K^2 \left(1 + \frac{1}{\gamma} \frac{m'(z)}{m(z)} \left(1 - \frac{\sigma_A}{\sigma_K} \varphi_{K,A} \right) \right). \tag{30}$$

Equation (30) shows that the market's equity premium is simply a function of the volatility of capital shocks, risk aversion, and the discount factor's (per unit of capital) sensitivity to changes in capital's productivity, $\frac{m'(z)}{m(z)}$. The equity risk-premium's expression is only indirectly a function of the elasticity of intertemporal substitution, though it is an explicit function of the volatility of capital shocks and risk-aversion.

The expression for human capital's risk-premium simplifies to:

$$r_{hc,t} - r_{f,t} \quad = \quad r_{e,r_f} \left(1 - \frac{h'(z_t)}{h(z_t)} \right) - \gamma \sigma_A^2 \left(\frac{\sigma_K}{\sigma_A} + \frac{1}{\gamma} \frac{m'(z_t)}{m(z_t)} \left(1 + \frac{\sigma_K}{\sigma_A} \varphi_{K,A} \right) \right) \frac{h'(z_t)}{h(z_t)}. \tag{31}$$

The return to human capital has two components. The first follows from human capital's correlation with the return on financial wealth, mitigated by the changes in production per unit of capital. This component is due to human capital's value being a linear function of the capital's stock; shocks to capital translate into shocks to the value of human capital. The shocks are mitigated by the fact that the value of the claim per unit of capital, $h(z_t)$, changes over time. The second component explaining the excess return on human capital stem from the impact that changes in production per unit of capital has on the way consumption is shared between dividends and wages. The numerical results shown below suggest this component is negative, *reducing* human capital's riskiness.

It is useful now to contrast the previous results with the returns to a claim to aggregate consumption. Recalling that the value of equity per unit of capital is one, and that the value of a claim to consumption equals the sum of claims to human capital and equity, the excess return for this claim will be given by:

$$\begin{aligned} r_{c,t} - r_{f,t} &= r_{e,r_f}\left(1 - \frac{h'(z_t)}{1+h(z_t)}\right) - \\ &\quad \gamma\sigma_A^2\left(\frac{\sigma_K}{\sigma_A} + \frac{1}{\gamma}\frac{m'(z_t)}{m(z_t)}\left(1 + \frac{\sigma_K}{\sigma_A}\varphi_{K,A}\right)\right)\frac{h'(z_t)}{1+h(z_t)}. \end{aligned} \tag{32}$$

Not surprisingly, the excess returns earned by consumption are *larger* than those earned by human capital but *smaller* than the excess returns to equity. Larger returns to equity can be achieved, in the presence of smooth consumption, through a process that prices human capital with lower returns than consumption. In other words, since consumption equals the sum of wages and dividends, if wages are less risky than consumption then dividends, through a "leverage" effect, must command a higher return premium. Research that includes labor income and attempts to calibrate asset-prices shares this result, though it typically does not make it explicit.

Equations (30) and (31) illuminate the differences between previous researcher conclusions about the returns to human capital. Baxter and Jermann (1997) do not define the claims to equity and human capital as I have here, and implicitly assume that $h'(z_t) = 0$. With this assumption, the returns to human capital and to equity are identical, and perfectly cointegrated. Lustig and Van Nieuwerburgh's (2006) results, however, imply that $h'(z_t) > 0$, so that innovations to equity returns have a negative covariance with innovations to human capital returns.

The relationship between wages and consumption determines the equity premium and the human capital premium. Whenever a negative shock hits the economy, consumption falls from its planned path and marginal utility increases. But wages do not fall us much as consumption or dividends, so the value of the stream of wages is relatively more valuable than the stream of dividends. The opposite is true after a positive shock, since marginal utility falls at the same time that dividends increase more than wages. The extra increase in dividends is not as valuable to the agent because dividend growth happens precisely when he values it less.

Finding the expected returns to equity and human capital allows us to answer the question: is human capital a bond or a stock? The calibrated model provides plausible answers. Now we can analyze these results using plausible parameters to evaluate what the model predicts about the variables described above. The results are discussed below.

4.1 The covariance of human capital and equity returns

Previous work (Jagannathan and Wang (1996)) used labor income growth as a proxy for human capital returns. Using this proxy they proceeded to test asset-pricing models and conclude that, given labor income growth's low covariance with equity returns, the impact that human capital has on asset-prices is small. This section studies the relationship between the covariance of labor income growth and stock returns and the covariance of human capital and stock returns to shed light on the relationship between each other.

The covariance between labor income growth and stock returns is

$$cov\left(\frac{dY_t}{Y_t}, \frac{dE_t}{E_t}\right) = \alpha\sigma_K^2 + (1-\alpha)\sigma_K\sigma_A\varphi_{K,A}, \tag{33}$$

whereas the covariance between human capital returns and stock returns is:

$$cov\left(\frac{dH_t}{H_t}, \frac{dE_t}{E_t}\right) = \left(1 - \frac{h'(z_t)}{h(z_t)}\right)\sigma_K^2 + \frac{h'(z_t)}{h(z_t)}\sigma_K\sigma_A\varphi_{K,A}. \tag{34}$$

Thus, the implications from the model is that the covariance between human capital and equity returns can be quite different than the covariance between labor income growth and equity returns. Given the common assumption of $\alpha \simeq .33$, if $\frac{h'(z_t)}{h(z_t)} \simeq .67$, then both the covariance of equity and human capital, and the covariance of equity and wages, would be similar. But the calibrated model suggests that this is not the case. The covariance between human capital and equity seems to be two or three times larger than the covariance between wage growth and equity.

The intuition for this result follows from the dynamics of the model. A positive shock leads to higher equity value and higher wages, implying a positive covariance between equity and wage growth. But wage growth is dampened relative to equity's return because following the positive shock wages are a smaller fraction of consumption. The return to human capital captures the fact that eventually wages will catch up dividends, so its covariance with equity is larger. In other words, returns to human capital capitalize wage growth, making human capital potentially more sensitive to the macro-economy.

5 Model calibration and results

We now study the behavior of the variables of interest in the model. In the analysis that follows, I will study the risk-free rate, the equity premium, the volatility of consumption and equity, the weight of human capital in the aggregate wealth portfolio and human capital's expected excess return.

5.1 Parameter choice

Table 1 shows the parameters I used to calibrate the model. I chose the parameters so that the model's long-term expected values match consumption growth, consumption volatility, production volatility, Sharpe ratios, the risk-free rate, and the ratio of wages to consumption observed in the data.

The technology's growth rate η determines consumption's long term growth. Thus I set η at 1.7% to match long-term consumption growth. The subjective discount rate ρ of 2% is consistent with other studies. The volatility in total factor productivity's growth rate, σ_A, is 2.5%. This value is consistent with the real business cycle literature, but I use it as a free parameter to help match the observed ratio of wages to consumption in the data. Capital's volatility, σ_K, poses a dilemma. On the one hand, in this model capital's volatility will also be equity's volatility. On the other hand, capital's volatility determines production volatility. Given the low volatility of production and the high volatility of equity observed in the data, the choice of σ_K will have to be a compromise between both. I choose 7%, consistent with an instantaneous volatility of production of 2.8%. The resulting volatility of stock returns of 7% is admitedly low, but it can be defended by noting that we observe levered equity, whereas in this model the appropriate volatility is the one of unlevered equity. Equity volatility of 7% is consistent with leverage slightly larger than 50%.[14]. Typical assumptions for capital intensity are between 30% and 40%, and I use 40% as the benchmark case. I choose the coefficient of relative risk aversion, γ (13), and the elasticity of intertemporal substitution, ψ (1.1), to match the risk-free rate and the market's Sharpe ratio and following other work using Epstein-Zin preferences. A relatively high value for γ ensures a high Sharpe ratio, whereas a low value for ψ produces low risk-free rates.

The depreciation rate δ and capital's intensity α have a large impact on the ratio of wages to consumption. Lower capital intensities are associated with higher ratios of wages to consumption and higher weights of human capital in the aggregate wealth portfolio. Higher depreciation rates have the same effect, increasing wages over consumption and the weight of human capital in the aggregate wealth portfolio. These variables do not significantly affect the other moments of the model. I choose α and δ to match the ratio of wages to consumption. The resulting parameter for δ is .06, which implies that capital investments have a half-life of roughly 11.5 years, and is common in the macroeconomics literature.

5.2 Results

The calibration provides several key insights. First, the ratio of wages to consumption is counter-cyclical, explaining why human capital is less risky than equity. Second, the size and risk of human capital are a function of the ratio of wages to consumption. Third, asset-prices change depending on the state of the economy. These results are discussed in

[14] Danthine and Donaldson (2002) use a similar argument to justify equity volatility of 12%

more detail below.

Using the parameters discussed in the previous section, Table 2 summarizes the values around which each of the variables mean-reverts. The risk-free rate of 1.7% is lower than the historical average. Equity excess returns are about 3.3% and instantaneous equity volatility is 7%. The resulting Sharpe ratio is .44. Equity's volatility is about two times higher than the volatility of consumption growth, which is 3.9%.

5.2.1 Consumption and output

First, we look at the ratio of consumption to capital. In the simple constant expected-returns model (for example, see Merton (1971)) consumption is a constant fraction of capital. In the economy analyzed here, consumption is adjusting as shocks change expected returns, resulting in a time-varying consumption to capital ratio. Figure 1 shows that the ratio of consumption to capital increases as a function of output per unit of capital. Since output per unit of capital increases after negative shocks, or at times in which future consumption growth is high, this result implies that consumption as a fraction of capital increases in "bad times."[15]

Next, we study the ratio of consumption to output. Figure 2 shows that this ratio decreases as output per unit of capital increases. In other words, the agent invests relatively more when the output for a given level of capital increases. This result is not surprising, since the agent's opportunity cost of consuming today increases when output per unit of capital is relatively high. Figure 2 also shows the ratio of wages to output. A property of the Cobb-Douglas production function is that this ratio is constant. As a result, consumption falls relative to wages when output per unit of capital increases. Output per unit of capital increases after shocks leave consumption below its long term value. These are times of relatively high marginal utility, implying that consumption falls more than wages after when the representative ageng is relatively "hungry". This is the key result explaining why human capital is less risky than equity in this model.

[15]Note that even though the ratio of consumption to capital increases in "bad" times, the absolute level of consumption falls, relative to its long run trend, in "bad times".

To vizualize the previous result, figure 3 presents the ratio of wages to consumption as a function of output per unit of capital. The figure shows that the ratio of wages to consumption increases with output per unit of capital. Because output per unit of capital is high when the representative agent is relatively “hungry”, this figure stresses the counter-cyclical movement of wages relative to consumption.

5.2.2 Human capital and equity returns

Figures 4 and 5 show how large and how risky human capital is. Unconditionally, human capital is 87% of aggregate wealth, and, at 2.3%, its expected excess return is about two thirds of equity's expected excess return. The weight of human capital in aggregate wealth is close to the fraction of wages to consumption, which has an unconditional value of 85%. Furthermore, the weight of human capital in aggregate wealth follows closely, and remains above, the ratio of wages to consumption.

Figure 5 shows the relationship between the ratio of wages to consumption and excess returns. Higher-than-average ratios of wages to consumption will be associated with higher-than-average contemporaneous excess returns. Equity's expected excess return is 3.3% and its volatility is 7%, as shown in figure 6. These values are driven by the “smoothing” induced by time-varying expected returns in the agent's consumption.

Besides human capital's size and expected return, Figure 7 shows the relationship between the covariance of aggregate wages growth and equity, and the covariance between human capital and equity. Throughout a wide range of values for production per unit of capital, including the steady-state, the covariance between human capital returns and equity returns is more than twice the covariance between wage growth and equity returns.

This result is relevant for asset-pricing tests and portfolio selection problems. After Mayer's (1972) result linking optimal portfolios and asset prices to aggregate wages growth, several papers (Fama and Schwert (1977), Heaton and Lucas (1996)), documented that since the observed covariance between wages growth and equity returns was small, the effect of human capital on asset-prices was small. But the model presented here shows that the covariance of human capital and equity returns is not equal to the covariance between wages growth and equity returns, with the covariance between human capital returns and equity returns appearing larger.

6 Conclusion

This paper explores the implications of a general equilibrium production model for the value and dynamics of human capital. Decreasing returns to scale and the sharing of consumption between labor and capital drive the results. The calibrated model predicts mean-reverting risk premia, dividend yields, and interest rates, with labor income growth and capital returns cointegrated over time.

The model suggests that human capital is less exposed to production shocks than equity. This result does not rely on frictions of labor markets such as labor contracts that protect workers from idiosyncratic productivity shocks. Instead, it follows from wages being a larger fraction of consumption in "bad times", so that owners of a human capital claim hold a natural hedge against unfavorable outcomes in the economy.

An empirical construction of the returns to human capital based on this model is better at explaining the cross-section of asset returns than labor income growth. This result opens the possibility for other tests that can be performed with this measure of the returns to human capital.

The model can be extended in multiple directions. An obvious one is allowing the fraction of output received by workers to change over time. Acemoglu (2002) presents a model grounded on new growth models from which the fraction of output received by capital is mean-reverting. Alternatively, even though capital intensity remains constant, the realized fraction could change due to "sticky" wages, for example due to the existence of adjustment costs. A richer model in which the fraction of output that goes to workers changes over time can magnify the "leverage" effect that wages have on dividends, as long as the fraction increases when consumption decreases. While in the present model uncertainty aligns the interests of investors and workers, translating into larger payoffs for both as technology changes, they are confronted by variations in the share of output they each receive. A larger pie benefits both stake holders, but the way the pie is divided clearly benefits one stakeholder at the expense of the other. Finding a way to separate these two effects is an extension that could help enhance our understanding of variations in labor income and asset returns.

Another extension is to consider heterogeneity in worker's skills and the inclusion of the dynamics of human capital accumulation. Distinguishing labor by its skill, and letting agents choose their level of skill could provide insights into our understanding of investments in human capital. Likewise, the model can be generalized to include demographic changes that would provide a link from intergenerational change to asset prices.

Finally, the role played by a competitive wage across different industries potentially affects the cross-section of asset returns. A positive technology shock in one industry will increase the demand for labor in that industry, which in turn increases economy-wide wages, affecting the returns in all other industries. Thus, labor markets play a significant role in the riskiness of firms' cash flows, and understanding that role better can improve our specification of the cross-section of asset returns.

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7 Appendix A: Symbols and variables

Variable	Description
α	Capital intensity.
A	Efficiency parameter in technology function
C	Consumption
c	Consumption per unit of capital $\left(\frac{C}{K}\right)$
d	Dividends per unit of capital
δ	Depreciation rate
$\varphi_{K,A}$	Correlation between technology and capital shocks
γ	Coefficient of risk aversion
J	Manager's value function
K	Aggregate capital
L	Aggregate Labor
S	Value of equity
e	Value of equity per unit of capital
H	Value of human capital
h	Value of human capital per unit of capital
ψ	Elasticity of Intertemporal Subsitution
β	Subjective discount rate
r_e	Expected return of equity
r_h	Expected return of human capital
r_f	Risk free rate
σ_A	Volatility of technology shocks
σ_K	Volatility of output shocks
σ_C	Volatility of consumption
σ_e	Volatility of of equity
Y	Labor income
y	Labor income per unit of capital
Z	Output per unit of capital
z	$\frac{1}{1-\alpha}\log Z$

8 Appendix B: Proofs

8.1 Proof of proposition 1

I start with the central planner's problem. Then, I solve for the manager's problem and show that it yields the same result as the central planner's problem.

The central planner's problem is:

$$J_t = \max_{\{c_t\}_t^\infty} E\left[\int_t^\infty f(C_s, V_s)ds\right] \tag{35}$$

$$s.t. \qquad dK_t = K_t(Z_t - \delta - c_t)dt + K_t\sigma_K dB_K, \tag{36}$$

where

$$f(C,V) = \frac{\beta}{1-\frac{1}{\psi}}(1-\gamma)V((C((1-\gamma)V)^{\frac{-1}{1-\gamma}})^{1-\frac{1}{\psi}} - 1). \tag{37}$$

Given the infinite-horizon setting, the solution will be time-independent. Duffie and SKiadas (1994) show that the solution to this problem can be solved using the traditional HJB equations, such that:

$$0 = f_C(C,J) + DJ, \tag{38}$$

where

$$DJ = \frac{\partial J}{\partial t} + \frac{\partial J}{\partial K}dK + \frac{1}{2}\frac{\partial^2 J}{\partial K^2}dK^2 + \frac{\partial J}{\partial A}dA + \frac{1}{2}\frac{\partial^2 J}{\partial A^2}dA^2 + \frac{\partial^2 J}{\partial K \partial A}dKdA \tag{39}$$

The structure of the problem suggests that J is homogeneous and, given the infinite-horizon, independent of t. Thus, guess the solution for J is given by:

$$J(K,z) = \frac{K^{1-\gamma}}{1-\gamma}g(z) \tag{40}$$

Calculating the FOC w.r.t. c and substituting the guess for $J(K, z)$, we obtain consumption per unit of capital as a function of $g(z)$:

$$c(z) = \beta^{\psi} \frac{g(z)^{\frac{1-\psi\gamma}{1-\gamma}}}{\left(g(z) - \frac{g'(z)}{1-\gamma}\right)^{\psi}}. \tag{41}$$

Next, replacing our guess for $J(K, z)$ in equation (39), and noticing that K drops out of the equation, we obtain the following ODE for the function $g(z)$:

$$\begin{aligned} 0 \;=\; & g(z)(1-\gamma)\left(\frac{\beta\psi}{1-\psi} + e^{(1-\alpha)z} - \frac{c(z)}{1-\psi} - \delta - \gamma\frac{\sigma_K^2}{2}\right) + \\ & g'(z)\left(-e^{(1-\alpha)z} + \frac{c(z)}{1-\psi} + \delta + \eta + \gamma\frac{\sigma_K^2}{2} - \gamma\frac{\sigma_A^2}{2} - (1-\gamma)\frac{\Omega^2}{2}\right) + \\ & g''(z)\frac{\Omega^2}{2} \end{aligned} \tag{42}$$

If a solution for $g(z)$ exists that satisfies equations (41) and (42), then our guess will be a solution to the central planner's problem. To verify that our candidate value function is indeed an optimal solution, we need to verify that the transversality condition holds. Numerical solutions imply the condition is satisfied, but I omit a formal proof here.

Now, find the solution to the competitive equilibrium. Following Cox and Huang (1989) and Duffie and Skiadas (1994), agent i's problem can be formulated as:

$$\max_{\{c_t\}_t^{\infty}} E\left[\int_t^{\infty} f(C_s, V_s)ds\right] \tag{43}$$

$$s.t. \qquad W_{i,t} \geq \int_t^{\infty} M_s(Y_s - C_s)ds, \tag{44}$$

where M_t is the stochastic discount factor. Taking the FOC w.r.t C, we find:

$$M_t = f_C(C_t, V_t)\lambda_i \tag{45}$$

Where λ_i is the lagrangian multiplier associated with agent i's optimization problem.

Firms have managers that take the stochastic discount factor M_t, wages Y_t, aggregate capital's level (K_t) and dynamics ($\mu_{K,t}$, σ_K) as given, and maximize the present value for shareholders. Specifically, firm j's manager solves the following problem:

$$\max_{\{L_j\}_t^\infty, \{d_j\}_t^\infty} E_t \left[\int_t^\infty M_s d_{s,j} ds \right] \tag{46}$$

$$s.t. \qquad dK_{j,t} = K_j(Z(K_{j,t}, L_{j,t}, A_t) - L_{j,t}y_t - d_t)dt + K_j \sigma_K dB_t$$

The firm's problem can be solved with dynamic programming. Defining the value function $J(K_{j,t}, A_t, K_t, t)$, the HJB of the firm's manager is:

$$0 = \sup_{L_{j,t}, d_{j,t}} M_t d_{j,t} + DJ \tag{47}$$

Where:

$$\begin{aligned} DJ \;=\; & \frac{\partial J}{\partial t} + \frac{\partial J}{\partial K} dK + \frac{1}{2}\frac{\partial^2 J}{\partial K^2} dK^2 + \frac{\partial J}{\partial A} dA + \frac{1}{2}\frac{\partial^2 J}{\partial A^2} dA^2 + \frac{\partial J}{\partial K_j} dK_j + \\ & \frac{1}{2}\frac{\partial^2 J}{\partial K_j^2} dK_j^2 + \frac{\partial^2 J}{\partial K \partial A} dK dA + \frac{\partial^2 J}{\partial K \partial K_j} dK dK_j + \frac{\partial^2 J}{\partial A \partial K_j} dA dK_j \end{aligned} \tag{48}$$

Using Equation (47), and noting that the demand for labor $L_{j,t}$ only appears in the dK_j term of DJ, the firm's FOC w.r.t $L_{j,t}$ produces:

$$L_{j,t} = \left(\frac{1-\alpha}{Y_t} \right)^{1/\alpha} K_{j,t} A_t^{(1-\alpha)/\alpha} \tag{49}$$

In equilibrium wages are such so that labor demand equals labor supply, i.e.,

$$L_t = \int_j L_j dj \tag{50}$$

$$= \left(\frac{1-\alpha}{Y_t} \right)^{1/\alpha} A_t^{(1-\alpha)/\alpha} \int_j K_{j,t} dj \tag{51}$$

$$= \left(\frac{1-\alpha}{Y_t}\right)^{1/\alpha} A_t^{(1-\alpha)/\alpha} K_t \tag{52}$$

Solving for Y_t:

$$Y_t = (1-\alpha)\left(\frac{K_t}{L_t A_t}\right)^{\alpha} A_t \tag{53}$$

Plugging into the firm's optimal demand for labor and back in the production function and taking into account depreciation, the dynamics of profits for firm j are:

$$K_{j,t} Z(K_j, L_j, A_t) - L_{j,t} Y_t - \delta K_{j,t} = K_j \left(\alpha \left(\frac{L_t A_t}{K_t}\right)^{1-\alpha} - \delta\right) \tag{54}$$

Using Equation (47), now consider the firm's FOC w.r.t to dividends:

$$M_t = \frac{\partial J}{\partial K_j} \tag{55}$$

Now we guess that the stochastic discount factor is only a function of aggregate capital K_t and the technology level A_t and time. Thus, from the firm's FOC we find that the value function for the manager is:

$$J(K_{j,t}, A_t, K_t, t) = K_{j,t} M(A_t, K_t, t) \tag{56}$$

Substituting into the HJB, noticing that the term with the dividend cancels out, and noticing that $K_{j,t}$ drops out of the equation we find:

$$\begin{aligned} 0 \;=\; & d_t M_t + \frac{\partial M_t}{\partial t} + M_t(\alpha e^{(1-\alpha)z_t} - \delta - d_t) + \\ & \frac{\partial M_t}{\partial A_t} A_t \eta + \frac{1}{2}\frac{\partial^2 M_t}{\partial A_t^2} A_t^2 \sigma_A^2 + \frac{\partial M_t}{\partial K_t} K_t \mu_K + \\ & \frac{1}{2}\frac{\partial^2 M_t}{\partial K^2} K^2 \sigma_K^2 + \frac{\partial M_t}{\partial A_t} A_t \sigma_K \sigma_A + \frac{\partial M_t}{\partial K_t} K_t \sigma_K^2 + \frac{\partial^2 M_t}{\partial A_t \partial K_t} A_t K_t \sigma_A \sigma_K \end{aligned} \tag{57}$$

Since all firms are subject to the same production shock, $\sigma_{K_j} = \sigma_K$. Also, note that K_j drops out of Equation (57), and we are left with an PDE that depends only on aggregate capital K_t, the technology level A and time. The solution to this PDE is the function for M, the stochastic discount factor, and the system will be solved. To continue with the solution, we guess that $M_t = \lambda K_t^{-\gamma} m(z_t) b(t)$, where $\lambda = \int_n \lambda_n dn$. Substituting in the previous equation, using equation 16), rearranging terms, and noticing that K_t drops from the equation, we find the ODE for the function $m(z_t)$,

$$
\begin{aligned}
0 \;=\; & m(z)\left(\frac{\beta(1-\gamma)\psi}{1-\psi} - \frac{1-\gamma\psi}{1-\psi} g(z)^{-1} c(z) m(z) + \alpha e^{(1-\alpha)z} - \delta - \gamma\mu_K - \frac{1}{2}\gamma(1-\gamma)\sigma_K^2\right) + \\
& m'(z)\left(\eta - \mu_K - \frac{\sigma_A^2}{2} + \frac{\sigma_K^2}{2} + (1-\gamma)(\sigma_A \sigma_K \varphi_{K,A} - \sigma_K^2)\right) + \\
& m''(z)\frac{1}{2}\Omega^2
\end{aligned}
\tag{58}
$$

The "no free-disposal" condition implies that consumption equals wages plus dividends. This implies that the drift of aggregate capital is:

$$
\mu_{K_t} = K_t e^{(1-\alpha)z_t} - C_t - K_t\delta \tag{59}
$$

Thus, Equation (58) becomes:

$$
\begin{aligned}
0 \;=\; & m(z)\left(\frac{\beta(1-\gamma)\psi}{1-\psi} - \frac{1-\gamma\psi}{1-\psi} g(z)^{-1} c(z) m(z)\right) + \\
& m(z)\left(\alpha e^{(1-\alpha)z} - \delta - \gamma(e^{(1-\alpha)z} - c(z)) - \frac{1}{2}\gamma(1-\gamma)\sigma_K^2\right) + \\
& m'(z)\left(\eta - (e^{(1-\alpha)z} - c(z)) - \frac{\sigma_A^2}{2} + \frac{\sigma_K^2}{2} + (1-\gamma)(\sigma_A \sigma_K \varphi_{K,A} - \sigma_K^2)\right) + \\
& m''(z)\frac{1}{2}\Omega^2
\end{aligned}
\tag{60}
$$

If the function $m(z)$ exists, then it simultaneously solves the manager's and agent's optimization problems. To verify that our candidate value function is indeed an optimal solution, we need to verify that the transversality condition holds. Numerical solutions suggest the condition is satisfied, but I omit a formal proof here.

8.2 Proof of proposition 2

I drop the subscript t where possible when it is not needed for clarity. The discounted process for the value of human capital is:

$$M_t H_t = E_t\left[\int_0^\infty M_{\tau+t} Y_{\tau+t} d\tau\right] \tag{61}$$

$$= E_t\left[\int_0^\infty M_{\tau+t}(1-\alpha)K_{\tau+t}e^{(1-\alpha)z_{\tau+t}} d\tau\right]. \tag{62}$$

The state variable z_t and the stock of productive capital K_t describe the economy, and so the value of human capital will only be a function of these two variables. Taking advantage of the homogeneity of the model, we guess that $H(K_t, z_t) = K_t h(z_t)$. Using this definition, and recalling that $M_t = \lambda K_t^{-\gamma} m(z_t) b(t)$, we can express 8.2 as:

$$K_t^{1-\gamma} m(z_t) b(t) h(z_t) = E_t\left[\int_0^\infty K_{t+\tau}^{1-\gamma} m(z_{t+\tau}) b(t+\tau)(1-\alpha)e^{(1-\alpha)z_{\tau+t}} d\tau\right]. \tag{63}$$

Using equation (16), the drift of the discounted process is:

$$\frac{E[dK_t^{1-\gamma} m(z)h(z)]}{K^{1-\gamma} m(z)h(z)} = \left(\frac{\beta(1-\gamma)\psi}{1-\psi} - \frac{1-\gamma\psi}{1-\psi} c(z)\frac{m(z)}{g(z)} - \frac{(1-\alpha)e^{(1-\alpha)z}}{h(z)}\right) dt. \tag{64}$$

Applying Itô-Doeblin's Lemma and taking expectations to the expression on the left hand side we obtain the following expression for the discounted process' drift:

$$\begin{aligned}\frac{E[dK^{1-\gamma} m(z)h(z)]}{K^{1-\gamma} m(z)h(z)} = {} & \mu_K(z) - r_f(z) + \sigma_K(\Sigma_2 + \Sigma_1\varphi_{K,A}) + \\ & + \frac{h'(z)}{h(z)}\left(\eta - \mu_K(z) - \frac{\Omega^2}{2} + \sigma_A(\Sigma_1 + \Sigma_2\varphi_{K,A}) - \sigma_K(\Sigma_2 + \sigma_1\varphi_{K,A})\right) \\ & + \frac{h''(z)}{h(z)}\frac{\Omega_K^2}{2}\end{aligned} \tag{65}$$

Equating both sides, the differential equation can be simplified to:

$$
\begin{aligned}
0 \;=\; & (1-\alpha)e^{(1-\alpha)z} + \\
& h(z)\left(-\frac{\beta(1-\gamma)\psi}{1-\psi} + \frac{1-\gamma\psi}{1-\psi}c(z)\frac{m(z)}{g(z)} + \mu_K(z) - r_f(z) + \sigma_K(\Sigma_2 + \Sigma_1\varphi_{K,A})\right) \\
& + h'(z)\left(\eta - \mu_K(z) - \frac{\Omega^2}{2} + \sigma_A(\Sigma_1 + \Sigma_2\varphi_{K,A}) - \sigma_K(\Sigma_2 + \Sigma_1\varphi_{K,A})\right) \\
& + h''(z)\frac{\Omega^2}{2}.
\end{aligned} \tag{66}
$$

If the function $h(z_t)$ exists, then it will satisfy Equations (64) and (66), and $H(K_t, z_t) = K_t h(z_t)$ is the arbitrage-free price of a claim to human capital.

Repeating the same process for a claim that pays aggregate dividends, the differential equation yields $S_t = K_t$ as a solution. Thus K_t is the value of a claim to equity in this economy.

9 Appendix C: Numerical solution method

The three ordinary differential equations (the first one solves for the function $g(z)$, the second one for $m(z)$, and the third one for $h(z)$) are found using the same method. I use the finite-difference Crank-Nicholson method as described in Marimon and Scott (1999) (chapter 8).

I first solve the ODE for the value function, using a grid of 1001 points for z ranging from -40 to 5, a wide range that captures the area of interest. The initial value is found from the closed-form solution when $z \to 0$. The initial guess is very important for the algorithm's convergence. I iterate until the maximum percent change for any given point is less than 1E-6.

Solving the ODE for the value of equity requires the value of $g(z)$ and $m(z)$ as inputs. The initial guess is with $h(z) = 0$, corresponding to the limit of $h(z)$ as $z \to -\infty$. I iterate until the maximum percent change between iterations is less than 1E-6.

Table I

Parameters used in calibration

This table shows the parameters used to calibrate the model. Section 5 describes the choice of each parameter in more detail.

Variable	Symbol	Value
Coefficient of relative risk aversion	γ	13
Intertemporal elasticity of substitution	ψ	1.1
Subjective discount rate	ρ	.02
Production shocks volatility	σ_K	7%
Productivity shocks volatility	σ_A	2.5%
Correlation between production and productivity shocks	$\varphi_{K,A}$	0%
Productivity growth	η	1.7%
Capital intensity	α	.40
Depreciation rate	δ	.06

Table II

Long-term trend values

This table records the values around which variables mean-revert in the calibration. Data on the risk-free rate, the risk-free premium, consumption growth volatility, equity volatility, and the Sharpe ratio is as reported in Campbell and Cochrane's (1999) long sample. The ratio of wages to consumption is the average for the period 1947 - 2007. Wages are equal to total compensation as reported in the National Income and Product Accounts (NIPA) tables. Consumption is the scaled sum of nondurables and services, also reported in the NIPA tables.

Variable	Model	Data
Risk-free rate	1.7%	2.9%
Equity risk premium	3.1%	3.9%
Consumption growth volatility	3.9%	3.8%
Equity volatility	7%	18%
Sharpe ratio	44%	28%
Wages over consumption	85%	85%
Human capital risk premium	2.3%	–
Human capital weight in aggregate wealth	87%	–

Table III

Sensitivity analysis

This table shows the sensitivity of the long-term values to changes in the parameters. The first three columns are parameters: α is capital's intensity, γ is the coefficient of relative risk aversion and σ_K is the volatility of production shocks. The middle columns show the results for the volatility of consumption growth and the ratio of wages to consumption w/c. The last columns show the results for the risk-free rate r_f, the equity risk premium r_e^e, the volatility of equity σ_e, equity's Sharpe ratio, and the excess return to human capital r_{hc}^e. Other parameters in the calibration are $\sigma_k = 0.07$, $\eta = 0.017$ and $\rho = 0.02$.

Parameters			Macro variables		Asset-prices				
γ	α	σ_A	w/c	σ_c	r_f	Sharpe	$r_{p,e}$	$r_{p,hc}$	w_{hc}
11	0.38	2.5%	0.948	3.9%	-0.18%	0.411	2.88%	2.08%	98.3%
11	0.38	3.5%	0.957	4.1%	-0.40%	0.415	2.91%	2.02%	99.0%
11	0.38	4.5%	0.969	4.3%	-0.67%	0.421	2.95%	1.93%	99.7%
11	0.4	2.5%	0.908	4.0%	0.57%	0.395	2.77%	2.03%	93.7%
11	0.4	3.5%	0.916	4.2%	0.34%	0.400	2.80%	1.97%	94.3%
11	0.4	4.5%	0.928	4.4%	0.04%	0.406	2.84%	1.88%	95.1%
11	0.42	2.5%	0.871	4.1%	1.27%	0.382	2.67%	1.99%	89.6%
11	0.42	3.5%	0.880	4.3%	1.01%	0.386	2.70%	1.93%	90.2%
11	0.42	4.5%	0.891	4.5%	0.69%	0.392	2.75%	1.84%	91.0%
13	0.38	2.5%	0.888	3.8%	0.78%	0.454	3.18%	2.37%	91.5%
13	0.38	3.5%	0.899	4.0%	0.43%	0.462	3.23%	2.30%	92.4%
13	0.38	4.5%	0.913	4.3%	-0.01%	0.471	3.30%	2.21%	93.5%
13	0.4	2.5%	0.852	3.9%	1.65%	0.436	3.05%	2.31%	87.4%
13	0.4	3.5%	0.862	4.1%	1.26%	0.444	3.10%	2.24%	88.3%
13	0.4	4.5%	0.876	4.3%	0.78%	0.453	3.17%	2.15%	89.3%
13	0.42	2.5%	0.819	4.0%	2.44%	0.420	2.94%	2.25%	83.8%
13	0.42	3.5%	0.829	4.2%	2.03%	0.428	2.99%	2.19%	84.6%
13	0.42	4.5%	0.843	4.4%	1.51%	0.437	3.06%	2.10%	85.6%
15	0.38	2.5%	0.835	3.7%	2.31%	0.485	3.39%	2.60%	85.3%
15	0.38	3.5%	0.847	3.9%	1.72%	0.496	3.47%	2.54%	86.4%
15	0.38	4.5%	0.863	4.2%	1.02%	0.510	3.57%	2.45%	87.8%
15	0.4	2.5%	0.802	3.8%	3.34%	0.464	3.25%	2.53%	81.7%
15	0.4	3.5%	0.813	4.0%	2.70%	0.475	3.33%	2.47%	82.7%
15	0.4	4.5%	0.829	4.3%	1.94%	0.490	3.43%	2.38%	84.1%
15	0.42	2.5%	0.772	3.9%	4.26%	0.446	3.12%	2.47%	78.4%
15	0.42	3.5%	0.783	4.1%	3.58%	0.458	3.20%	2.41%	79.4%
15	0.42	4.5%	0.798	4.3%	2.75%	0.472	3.30%	2.33%	80.7%

Figure 1

Consumption per unit of capital as a function of output per unit of capital

Results with the following parameters: $\alpha = 0.40$, $\rho = 0.02$, $\eta = 0.017$, $\gamma = 13$, $\psi = 1.1$, $\sigma_K = 0.07$, $\delta = .06$. The vertical black line denotes the long-term value around which the economy oscillates.

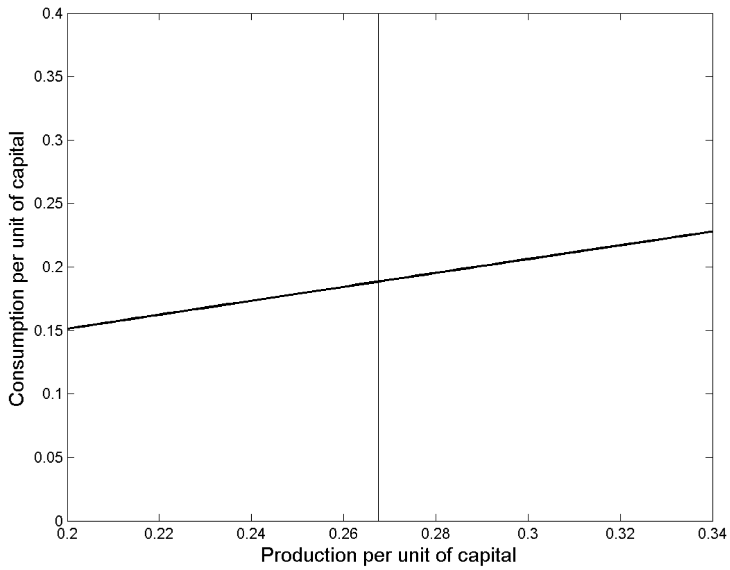

Figure 2

Consumption over output as a function of output per unit of capital

Results with the following parameters: $\alpha = 0.40$, $\rho = 0.02$, $\eta = 0.017$, $\gamma = 13$, $\psi = 1.1$, $\sigma_K = 0.07$, $\delta = .06$. The vertical black line denotes the long-term value around which the economy oscillates.

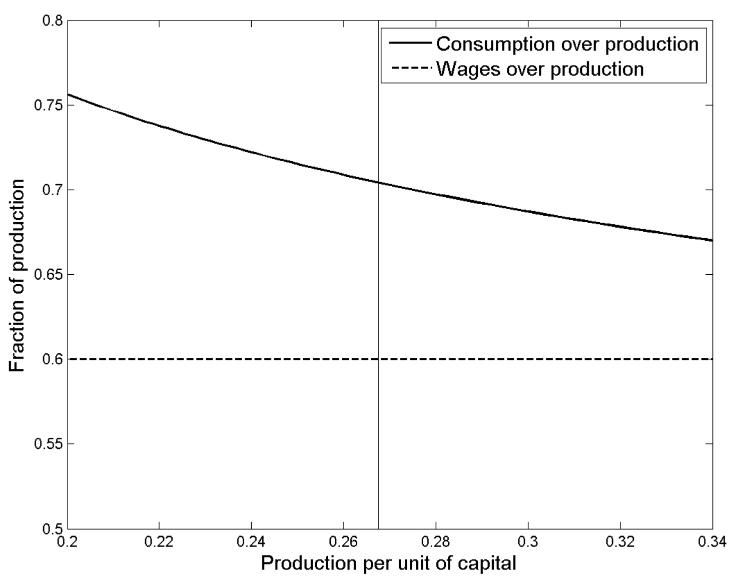

Figure 3
Wages over consumption as a function of output per unit of capital

Results with the following parameters: $\alpha = 0.40$, $\rho = 0.02$, $\eta = 0.017$, $\gamma = 13$, $\psi = 1.1$, $\sigma_K = 0.07$, $\delta = .06$. The vertical black line denotes the long-term value around which the economy oscillates.

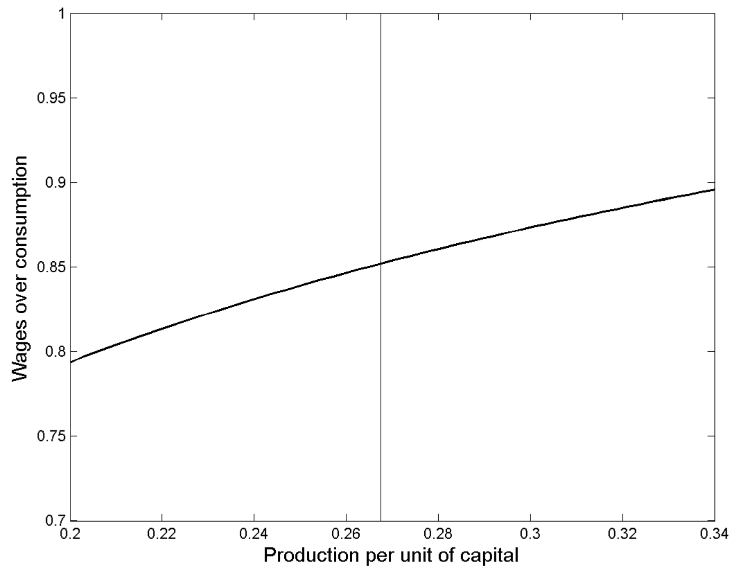

Figure 4
Human capital as a fraction of the wealth portfolio

Results with the following parameters: $\alpha = 0.40$, $\rho = 0.02$, $\eta = 0.017$, $\gamma = 13$, $\psi = 1.1$, $\sigma_K = 0.07$, $\delta = .06$. The vertical black line denotes the long-term value around which the economy oscillates.

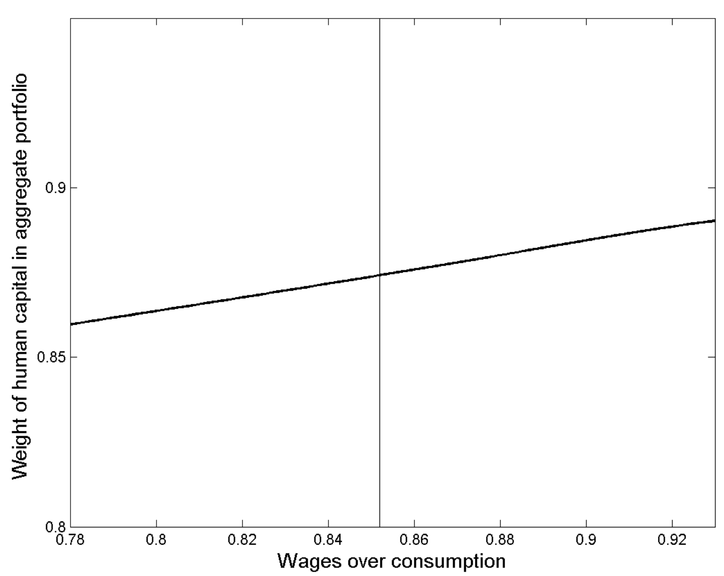

Figure 5
Expected risk premium and as a function of wages over consumption

Results with the following parameters: $\alpha = 0.40$, $\rho = 0.02$, $\eta = 0.017$, $\gamma = 13$, $\psi = 1.1$, $\sigma_K = 0.07$, $\delta = .06$. The vertical black line denotes the long-term value around which the economy oscillates.

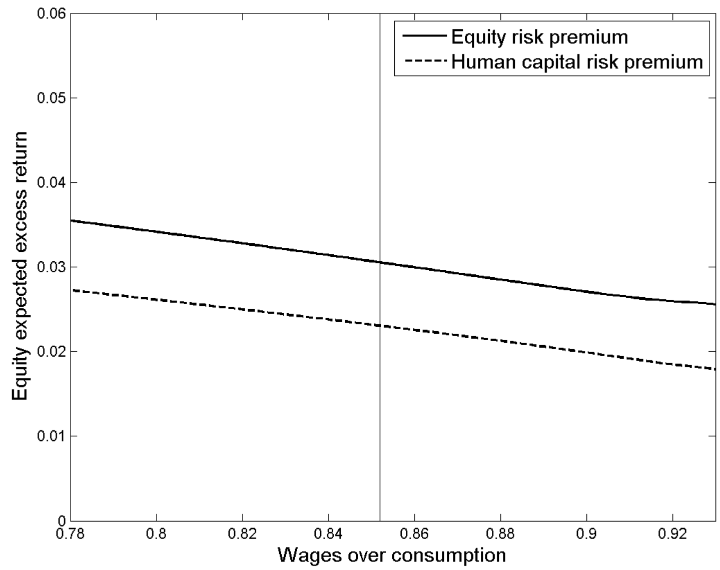

Figure 6
Consumption and equity volatility as a function of wages over consumption

Results with the following parameters: $\alpha = 0.40$, $\rho = 0.02$, $\eta = 0.017$, $\gamma = 13$, $\psi = 1.1$, $\sigma_K = 0.07$, $\delta = .06$. The vertical black line denotes the long-term value around which the economy oscillates.

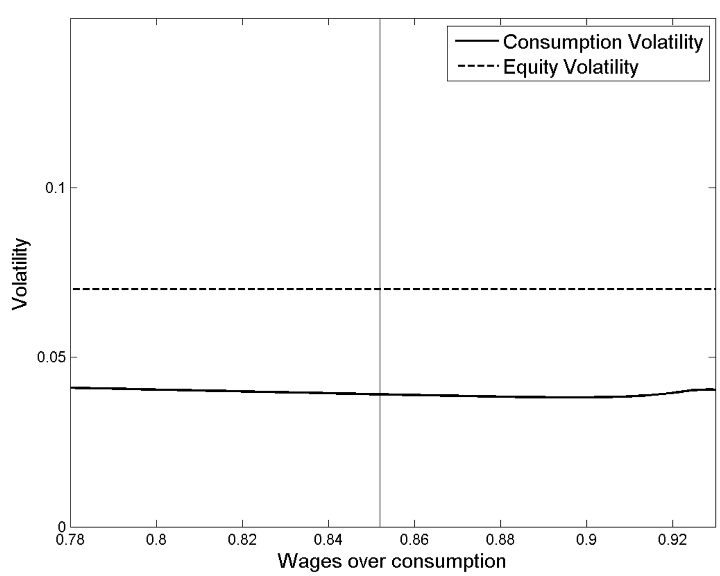

Figure 7

Covariance between wages, human capital, and equity returns

Results with the following parameters: $\alpha = 0.40$, $\rho = 0.02$, $\eta = 0.017$, $\gamma = 13$, $\psi = 1.1$, $\sigma_K = 0.07$, $\delta = .06$. The vertical black line denotes the long-term value around which the economy oscillates.