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ENDOGENOUS SOCIAL NETWORKS AND INEQUALITY IN AN INTERGENERATIONAL SETTING

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Abstract

The paper offers a novel view of the consequences for inequality of the joint evolution, endogenous or exogenous, of social connections and human capital investments. It allows for intergenerational transfers of both human capital and social networking endowments in dynamic and steady-state settings of dynastic overlapping-generations models of increasing demographic complexity. Intergenerational transfer elasticities exhibit rich dependence on social effects. The separable effects on human capital dispersion of social interactions alone, as distinct from the joint effects with the intertemporal evolution of skills, are analyzed. The dynamics of demographically increasingly complex models are shown to be tractable. Their stochastic steady states allows us to study the cross-section human capital distribution in the presence of shocks to underlying parameters that may be interpreted as shocks to cognitive and non-cognitive skills.

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1 Introduction

In a world where individuals interact in myriads of ways, one wonders how the benefits of one's connections with others compare with those conferred by individual characteristics when it comes to acquisition of human capital. It is particularly interesting to be able to distinguish between connections that are the outcome of deliberate decisions by individuals and connections being given exogenously and beyond individuals' control. Such a distinction matters macroeconomically as well, if individuals stand to benefit from social connections in ways that affect consumption and investment. Individuals may seek to form social links with others, as an objective in its own right, in order to enrich their social lives and avoid social isolation. Social links provide conduits through which benefits from interpersonal exchange can be realized. Social isolation excludes them. The paper explores the consequences of the joint evolution of social connections and human capital investments. It thus allows one to study the full extent in which social connections may influence inequality in consumption, human capital investment and welfare across the members of the economy. It embeds inequality analysis in models of endogenous social networks formation. The novelty of the model lies in its joint treatment of human capital investment and social network formation, while distinguishing between the case of impact on human capital from endogenous as opposed to exogenous social networking.

The last few years have generated new research on social networks at a torrential rate, including books, most notably Goyal (2009) and Jackson (2008), and hundreds of papers. Social networks research was booming within econophysics for more than twenty years while being hardly noticed by economists. Nowadays, social network research is increasingly spreading to virtually all economics fields, including notably experimental economics, too. Yet, as Jackson (2014), p. 14, points out, studying endogenous network formation continues to be an important priority. The present paper aims at a deeper understanding of the consequences of social network formation for inequality. Such an emphasis has an intuitive appeal, that is whether social networking increases or decrease inequality.

It is straightforward to assess the difficulty of modeling social networking. For a given number of individuals I, there are $2^{\frac{I(I-1)}{2}}$ different possible networks connecting them. Thus, to a typical social group of $I = 100$ there correspond $2^{50\times49} \approx 10^{1500}$ network configurations, some of which are not topologically distinct. As Blume, Brock, Durlauf and Jayaraman (2015) argue, there is no viable general theoretical model of network formation. Therefore, to be able to conduct specific analyses that link differences in personal characteristics to differences in outcomes after individuals have formed social networks, and have been influenced by those they end up being in social contact with, one needs to be specific. It is for this reason that we start with a fairly tractable model of social network formation, which is due to Cabrales, Calvó-Armengol and Zenou (2011), which we extend into a dynamic model.[2]

The Cabrales, Calvó-Armengol, and Zenou framework originally starts from a familiar linear-quadratic model of individual decision making, based on connecting with others in a multi-person group context, with social links seen as outcomes of individual decisions, which are associated with a noncooperative Nash equilibrium.[3] A connection between any two individuals is associated with a connection weight, whose magnitude depends on inputs of effort by the two respective individuals, which can be either exogenous or functions of inputs decided upon by the respective parties. The results are obtained in a framework where links are symmetric (i.e., the underlying graph is undirected but weighted) and thus the benefits are mutual. The formation of undirected (symmetric) links, as modelled here, presumes a certain degree of social coordination. That is, individuals recognize that even though their decisions are made in a non-cooperative context, they nonetheless result in social group formation. Asymmetric links, as where my being influenced by others (as by looking up to others) does not presume that those other individuals I am linked to are in turn influenced by me, provide avenues of social influence but do not connote social relations as such.

This paper extends the Cabrales *et al.* model so as to allow for cognitive shocks in a static context assuming a CES interactions structure. It then extends further by means of a number of dynamic models of human capital investment and social network formation in order to allow for intergenerational transfers of wealth and of social connections. First, we interpret the dynamic model as one with the representative individual being infinitely lived. A variation of that model is to take social connections as given exogenously and not subject to optimization. This variation allows us to highlight the importance of endogenous setting of social connections for the cross-sectional distribution of human capital and explore conditions under which the social connections help magnify or reduce the impact of the dispersion in cognitive skills. When social connections are endogenous multiple equilibria become possible. At the steady state solutions associated with either high or, alternatively low socialization efforts, the distribution of human capital mirrors that of the cognitive skills. Next, we follow a long tradition in economics that links life cycle savings, human capital investment and intergenerational transfers. Starting from Loury (1981), but also Becker and Tomes (1979)[4], a number of papers have linked intergenerational transfers and the cross-section distributions of income and of wealth. In a recent paper, Lee and Seshadri (2014) model human capital accumulation in the presence of intergenerational transfers, while allowing for multiple stages of investment over the life cycle, such as investment during childhood, college decision and on-the-job human capital accumulation. Theirs is one of very few papers that take Heckman's forceful suggestion [see Cunha and Heckman (2007); Heckman and Mosso (2014)] seriously, namely to allow for complementarity between early and later child investments, *inter alia,* by means of a model of 78-overlapping generations (and thus many more than the commonly used two overlapping generations) with infinitely lived altruistic dynasties. Their model shows, using numerical simulation methods, that investment in children and parents' human capital have a large impact on the equilibrium intergenerational elasticities of lifetime earnings, education, poverty and wealth, while remaining consistent with cross-sectional inequality. They also show that education subsidies and progressive taxation can significantly reduce the persistence in economic status across generations. But they do not model social connections.

There is a long-standing empirical literature on different aspects of intergenerational mobility across different countries. Corak (2013) emphasizes an empirical pattern, known as the "The Great Gatsby Curve:" higher earnings inequality is associated with lower intergenerational mobility. Black and Devereux (2011) survey the key developments regarding the forces driving the correlations between earnings among successive generations. Black, Devereux and Salvanes (2009) report estimates of the intergenerational transmission of IQ scores: an increase in father's IQ at age 18 of 10% is associated with a 3.2% increase in son's IQ at the same age. While most empirical research focuses on the persistence of income or of economic status across two successive generations, recent research has ventured into persistence across up to four successive generations. In particular, Lindahl *et al.* (2015) obtain estimates, using Swedish data, of intergenerational transmission of individual measures of lifetime earnings for three generations and of educational attainment for four generations. They find that estimates obtained from data on two generations severely underestimate long-run intergenerational persistence in both labor earnings and educational attainments. This in turn implies that much lower long-run social mobility in terms of dynastic human capital, which they attribute to direct influence across generations by more distant family members than parents. Specifically, the directly estimated coefficients by means of a single regression of the great-grandparent's education on that of the grandparent is 0.607, on that of the parent 0.375 and on that of the child is 0.175. Similarly, the estimated coefficient of the grandparent's earnings on that of the parent is 0.356, and on that of the child is 0.184. These are much larger than those imputed from conventionally estimated correlations of the respective magnitudes between two successive generations.

Black *et al.* (2015) seek to separate the impact of genetic from environmental factors as determinants of the intergenerational transmission of net wealth by means of administrative data for a large sample of Swedish adoptees merged with similar information for their biological and adoptive parents. Comparing the relationship between the wealth of adopted and biological parents and that of the adopted child, they find that, even prior to any inheritance, there is a substantial role for environment and a much smaller role for genetics. In examining the role of bequests, they find that, when they are taken into account, the role of adoptive parental wealth becomes much stronger. Their findings suggest that wealth transmission is not primarily because children from wealthier families are inherently more talented or more able but that, even in relatively egalitarian Sweden, "wealth begets wealth."

Specifically, the effect on the child rank in within-cohort wealth distribution of the rank of biological parent wealth has an estimated coefficient of 0.162 and that of the adoptive parent wealth of 0.222, while those for inheritance are 0.124 and 0.231, respectively, all very highly significant statistically. These findings are also corroborated by Englund *et al.* (2013), who use administrative data from Sweden that follow a panel of parents matched to their grown children. They find that childrens initial endowments of net worth and their subsequent net worth accumulations are positively correlated with parents' net worth, and that children of wealthy parents have higher earnings, even conditional on intergenerational correlation in earnings. They argue that the intergenerational correlation in net worth comes largely from housing wealth, which they explain in terms of correlations in home ownership among high net worth parents and their children, as well as a number of other factors.

Clark (2014) has also contributed to revival of interest in the persistence of status over long periods of time and the reasons for it. Using surnames to track generations, Clark shows that true rates of social mobility are much slower than conventionally estimated. Furthermore, they are not any higher now than in the pre-industrial era, and they vary surprisingly little across societies. Social mobility rates are as slow in egalitarian Sweden as they are in inegalitarian Chile. Clark's findings pose awkward questions about whether social policy can do much to increase the rate of regression to the mean of "elites and underclasses." Grönquist et al. (2014) report that the intergenerational correlation between fathers and sons, obtained from Swedish records of military enlistment for 37 cohorts range in 0.42–0.48 for cognitive, and around 0.42 for non-cognitive abilities. Their results show that mother-son correlations in cognitive abilities are somewhat stronger than father-son correlations, while no such difference is apparent for non-cognitive abilities. Furthermore, to the intergenerational transmission from fathers to sons of cognitive skills, non-cognitive skills also contribute in a statistically significant way, but with a numerically much smaller coefficient, 0.445 vs. 0.069; and correspondingly, of non-cognitive skills, cognitive skills also contribute in a statistically significant way, but with a numerically much smaller coefficient, 0.043 vs. 0.391. Such effects are strengthened by assortative mating; see Güell *et al.* (2015).

The present paper relies on these estimates as a source of motivation to study the role of both cognitive skills, which are enhanced with education and training, along with non-cognitive skills, which are more closely related to social networking. Thus, the dynamics of human capital accumulation may be jointly studied with the evolution of social connections. It presents a sequence of models, with parents making decisions about how much wealth to transfers to the children and about social connections along with investment in human capital. Parents recognize that due to the timing of implementing their social networking decisions their children stand to benefit from them, as they themselves have benefited from the decisions of their own parents. By moving to a model with two overlapping generations, we can determine how the pattern of dynamics reflects the demographic structure of the economy. Furthermore, as the number of overlapping generations increases, the matrix characterizing the dynamic evolution of the state variable has a multiplicative factorial structure: each additional overlapping generation included contributes a factor to the product. Finally, the paper examines a variation of the two overlapping generations model with two subperiods which makes it possible for individuals to invest in augmenting the cognitive skills of their children. The impact of availability of such investments on the dynamics of evolution of human capital investments and social connections is considerably more complicated, but a factorial structure is still evident.

The remainder of this document is organized as follows. Section 2 introduces the basic model in a static setting. This model allows us to explore the empirical implications of endogeneity of social connections by allowing for different assumptions about the effects of interactions. While the value of interactions and their consequences for income inequality have been explored before, notably by Benabou (1996) and Durlauf (1996; 2006), those earlier analyses do not allow for social network formation. Next we use the model to explore the case when each individual's interactions with her social contacts are of the CES-type, as an example of many alternative specifications. Section 2.2.1 introduces shocks to one of individuals' behavioral parameters that I interpret as shocks to cognitive skills. Section 3 presents an infinite-horizon model of an evolving economy consisting of many agents who build connections among each other. Section 4 assesses some consequences for cross-sectional inequality. Section 5 interprets the model in an infinite-horizon dynastic life cycle context, and section 5.1 extends the model first to an overlapping generations context, ultimately with two-overlapping generations. Subsection 5.1.2 examines, in particular, the effects of social networking on intertemporal wealth transfer elasticities, and subsection 5.3 introduces shocks to another of individuals' behavioral parameters that I interpret as shocks to non-cognitive skills. The solution allows us to discuss the properties of models with more than two overlapping generations. These extensions allow for parents' circumstances to influences their children's wealth endowments via transfers, social networking, as well as possibly persistent cognitive skills.

2 Endogenous Social Structure: The Cabrales, Calvó-Armengol and Zenou Model

In commonly employed formulations of models of individuals' actions subject to social interactions and in the definition of the group choice problem each individual is typically assumed to be affected by group averages of contextual effects and of decisions [Ioannides 2013, Ch. 2]. It is easy to contemplate that individuals may deliberately seek social interactions that are not necessarily uniform across their social contacts and to examine their determinants. In the absence of a "viable general theoretical model of network formation" [Blume, Brock, Durlauf and Jayaraman (2015), p. 474] I adopt the Cabrales, Calvó-Armengol and Zenou (2011) as a parsimonious starting point. Immediately below, I briefly develop their key results, with individuals' engaging in networking efforts (socialization, in their terminology) that determine the probabilities of contacting others simultaneously while deciding on their own actions. Further below, I interpret individuals' actions as human capital investments.

Individual i chooses action k_i and socialization effort s_i, taking as given actions and socialization efforts by all other individuals, $i, j \in \mathcal{I}$, so as to maximize:

$$U_{i,\tau(i)}(\mathbf{s}, \mathbf{k}) \equiv b_{\tau(i)} k_i + a \sum_{j=1, j\neq i}^{I} g_{ij}(\mathbf{s}) k_i k_j - c\frac{1}{2}k_i^2 - \frac{1}{2}s_i^2, \tag{1}$$

where $\tau(i)$ denotes the individual type[5] individual i belongs to. I will simplify this notation for clarity, when it is not necessary, by using i instead of $\tau(i)$. The terms $\mathbf{s} =$

$(s_1, \ldots, s_i, \ldots, s_I)$ denote the full vector of networking efforts, and $\mathbf{k} = (k_1, \ldots, k_i, \ldots, k_I)$, those of actions. The weights of social interaction g_{ij}, the elements of a social interactions matrix $\mathbf{G}$, may be defined in terms of socialization efforts in a number of alternative ways. In the simplest possible case, let the weights, which are obtained axiomatically by Cabrales *et al.,* be defined as:

$$g_{ij}(\mathbf{s}) = \frac{1}{\sum_{j=1}^{I} s_j} s_i s_j, \text{ if } \forall s_i \neq 0; \quad g_{ij}(\mathbf{s}) = 0, \text{otherwise.} \tag{2}$$

The coefficient of the interactive term in definition (1) is a key parameter in the determination of $\mathbf{s}$, the vector of connection intensities. Individual i chooses (s_i, k_i) so as to maximize (1).

I follow Cabrales *et al.* (2011) and define, for later use, an auxiliary variable

$$\tilde{a}(\mathbf{b}) = a \frac{\sum_{\tau \in \mathcal{T}} b_\tau^2}{\sum_{\tau \in \mathcal{T}} b_\tau}, \tag{3}$$

where $\mathcal{T}$ denotes the set of agent types, with generic element τ, as distinct from the set of individuals, $\mathcal{I}, I = |\mathcal{I}|$, and the functions $\bar{x}(\mathbf{x}), \overline{x^2}(\mathbf{x})$ are defined as follows:

$$\bar{x}(\mathbf{x}) \equiv \frac{\sum_{\tau \in \mathcal{T}} x_\tau}{|\mathcal{T}|}, \ \overline{x^2}(\mathbf{x}) \equiv \frac{\sum_{\tau \in \mathcal{T}} x_\tau^2}{|\mathcal{T}|}. \tag{4}$$

The normalized sums in this definition reflect relative frequencies of individual types.

The first-order conditions are, with respect to k_i, s_i, as follows:

$$b_{\tau(i)} + a \sum_{j=1, j \neq i}^{I} g_{ij}(\mathbf{s}) k_j - c k_i = 0; \tag{5}$$

$$a \sum_{j=1, j \neq i}^{I} k_i k_j \frac{\partial g_{ij}(\mathbf{s})}{\partial s_i} - s_i = 0. \tag{6}$$

With $g_{ij}(\mathbf{s})$ given by (2),

$$\frac{\partial g_{ij}(\mathbf{s})}{\partial s_i} = \frac{1}{\sum_{j=1}^{I} s_j} s_j - \frac{1}{(\sum_{j=1}^{I} s_j)^2} s_i s_j.$$

Following Ballester *et al.* (2006) and Cabrales *et al.* (2011), it is convenient to rewrite the first-order conditions, respectively, as follows:

$$\left[\mathbf{I} - \frac{a}{c} \mathbf{G}(\mathbf{s})\right] \cdot c\mathbf{k} + a \ \mathrm{diag}\left(\mathbf{G}(\mathbf{s})\right) \cdot \mathbf{k} = \mathbf{b}. \tag{7}$$

As they note, the matrix $[\mathbf{I} - \frac{a}{c}\mathbf{G}(\mathbf{s})]$ is invertible and has a particularly simple form, using which (7) becomes:

$$c\mathbf{k} + a[\mathbf{I} + \lambda_{a/c}(\mathbf{s})\mathbf{G}(\mathbf{s})] \cdot \operatorname{diag}(\mathbf{G}(\mathbf{s})) \cdot \mathbf{k} = [\mathbf{I} + \lambda_{a/c}(\mathbf{s})\mathbf{G}(\mathbf{s})] \cdot \mathbf{b}, \tag{8}$$

where $\lambda_{a/c} \equiv \frac{a}{c}\frac{\bar{x}(\mathbf{s})}{\bar{x}(\mathbf{s}) - \frac{a}{c}\overline{x^2}(\mathbf{s})}$. Rewriting (6), the first-order conditions for the s_i's, yields:

$$s_i = ak_i \frac{\mathbf{s} \cdot \mathbf{k}}{I\bar{x}(\mathbf{s})} - as_i k_i \frac{\mathbf{s} \cdot \mathbf{k}}{(I\bar{x}(\mathbf{s}))^2} - a\frac{s_i k_i}{I\bar{x}(\mathbf{s})} + a\frac{(s_i k_i)^2}{I\bar{x}(\mathbf{s})^2}, \tag{9}$$

where $\mathbf{s} \cdot \mathbf{k} = \sum_{i=1}^{I} s_j k_j$, while abusing notation and identify summing over types with summing over individuals.

2.0.1 Solving with a Large Number of Agents

As $I \to \infty$, the last three terms on the RHS of (9) vanish. Such simplifications via limiting results as $I \to \infty$, which Cabrales et al. (2011) make use of repeatedly, recur throughout the present paper and will not be derived de novo each time they are invoked. Thus, (9) becomes:

$$s_i = ak_i \frac{\mathbf{s} \cdot \mathbf{k}}{I\bar{x}(\mathbf{s})}. \tag{10}$$

Similarly, since $g_{ii}(\mathbf{s}) = \frac{s_i^2}{\sum_{j=1}^{I} s_j}$, $\operatorname{diag}(\mathbf{G}(\mathbf{s}))$ vanishes at the limit, as I becomes large. Thus (8) becomes:

$$c\mathbf{k} = [\mathbf{I} + \lambda_{a/c}(\mathbf{s})\mathbf{G}(\mathbf{s})] \cdot \mathbf{b}. \tag{11}$$

2.1 The Cabrales, Calvó-Armengol and Zenou Benchmark Case

Having demonstrated in some detail the approach pioneered by Cabrales *et al.* (2011), I next summarize their key result by means of the following proposition. For a proof, See Appendix A, Proofs.

Proposition 1. When the number of agent types I is large:

Part a. The multi-person game admits the solution $s_i = 0$, $k_i = \frac{1}{c}b_i$, which will be referred to as autarkic, and optimal actions and socialization efforts are given by:

$$k_i = \vartheta b_i, \; s_i = \varpi\vartheta b_i, \tag{12}$$

where ϖ, ϑ Satisfy the system of algebraic equations:

$$\varpi = \tilde{a}\vartheta; \tag{13}$$

$$\vartheta = \frac{1}{c - \varpi^2}. \tag{14}$$

Part b. The system of equations (13–14) admits two sets of positive solutions, provided that:

$$2\left(\frac{c}{3}\right)^{\frac{3}{2}} \geq \tilde{a}. \tag{15}$$

If agents do not value connections, $a = 0$, they do not exert socialization efforts, and from (3) and (13), $\varpi = 0$. The autarkic solution follows: $k_{i,\text{aut}} = \frac{1}{c} b_{\tau(i)}$. If agents do exert socialization efforts, that is they are connected, $\varpi > 0$, and $\vartheta^* > \frac{1}{c}$, and the k_i's exceed their autarkic values. It is for this reason that Cabrales *et al.* refer to the equilibrium values of ϑ as the social (synergistic) multiplier. Exerting socialization efforts and acquiring social links provide incentives that lead to increased human capitals and improved individual welfare. The feasibility condition for a non-autarkic solution, which Cabrales *et al.* obtain,[6] readily follows from the closed form solution of the cubic equation. For at least one solution to exist, from (15), the magnitude of $\tilde{a}$, the social interactions coefficient adjusted by the excess dispersion of the individual cognitive skills, must not exceed a function of the marginal cost of action coefficient. If the above condition is satisfied with inequality, then two solutions exist, leading to a high and a low equilibrium, in the terminology of Cabrales *et al.*, both of which are stable and Pareto-rankable. The socially efficient outcome lies between those two equilibria. Restrictions on parameter a on account of feasibility recur throughout the paper.

Numerous alternative formulations for the interaction structure are possible. A number of alternative formulations are examined in an unpublished earlier version of Cabrales *et al.* (2011), where the terms $g_{ij}(\mathbf{s})$ are specified as $g_{ij}(\mathbf{s}; \mathbf{b})$. Notably, such an analysis demonstrates the significance of homogeneity of degree less than or equal to one in connection weights. If that degree exceeds one, then because of too many synergies, as I grows, the social structure becomes infeasible.[7]

2.1.1 Eliminating Equilibrium Multiplicity

As indicated above, Cabrales *et al.* prove that both high and low equilibria are stable and Pareto-rankable. Equilibrium mutliplicity may be attractive in certain settings, because it is known that outcomes may differ across communities that otherwise exhibit quite similar fundamentals.[8] Yet, it might be awkward in a macro context. One way to remove the awkwardness is to embed the model in a macro framework. We can introduce an output sector that produces a good using only labor as a input under constant returns. Individuals supply their labor in the form of efficiency units, with aggregate supply being given by $\sum_i b_i h_i$, and are remunerated by means of a constant wage rate, ω. At equilibrium, profits are zero. The fact that the high equilibrium is Pareto superior to the low equilibrium, allows us to devise a subsidy and tax scheme $(\xi(s_i), S(s_i))$ so as to induce each individual to supply socialization effort and invest in human capital consistent with the high equilibrium. Let the wage subsidy be a function of s_i, $\xi(s_i)$, so that gross wage income would be $(1+\xi(s_i))\omega b_i h_i$. Optimizing with respect to s_i leads individual i to exert greater effort and correspondingly invest more in human capital. An associated lump-sum tax can leave each individual at a desirable level of net income. The parameters of the subsidy and tax scheme may be chosen so as the subsidy and tax scheme be revenue neutral, while inducing individuals to choose human capital and socialization effort associated with the high level equilibrium. This argument applies equally well to the dynamic settings where equilibrium multiplicity emerges when social networking is endogenous, which we investigate further below, and will not be repeated.

2.2 CES Interactions Structure

Social interaction weights may be generalized so as to involve the k_i's and thus express complementarity effects. Here we explore a CES interactions structure of homogeneity of degree one, but it is also possible to allow for individual characteristics to influence weights in a great variety of ways, for homogeneity of degree other than in interactions weights. For example, the terms $g_{ij}(\mathbf{s};\mathbf{b})k_i k_j$, express synergy weights between agents i and j. The marginal utility of human capital k_i depends positively on those of other agents via a convex structure. We generalize this assumption by means of a CES structure,[9] which may be either convex or concave in the inputs. It is well known that in the limit, such a structure allows for an individual to benefit from the *maximum* or the *minimum*, respectively, among all other individuals he interacts with [Benabou (1996); Polya *et al.* (1952), p. 15, Theorem 4].

That is, if the interaction term in (2) may be assumed to be instead of the form:

$$k_i s_i \left[\sum_{j \neq i} \frac{s_j}{\sum_i s_i} k_j^{1-\frac{1}{\xi}} \right]^{\frac{\xi}{\xi-1}}, \tag{16}$$

then it admits as a special case the original assumption (2), as well as a number of commonly used assumptions as additional special cases. That is, special cases of (16) are notable: 1. $\xi \to \infty : k_i \ s_i \frac{s_j}{\sum_i s_i} k_j$; 2. $\xi \to 1 : k_i s_i \prod_{j \neq i} k_j^{\left(s_j / \sum_i s_i\right)}$; 3. $\frac{1}{\xi} \to \infty : \min_j \{k_j\}$: *one bad apple spoils the bunch.* 4. $\frac{1}{\xi} \to -\infty : \max_j \{k_j\}$: *the best individual is the role model.*

Case 1 coincides with the original specification in Section 1 above. Case 2 is the classic Cobb-Douglas function as special case of the CES structure; case 3 is the Leontieff case; case 4 is the extreme case of a convex interaction structure.[10] Next I examine first the deterministic case of special cases 3 and 4 above and then the uncertainty counterparts with stochastic cognitive shocks. For a proof, see appendix A, Proofs.

Proposition 2. *For the CES interactions assumption (16), under* $\frac{1}{\xi} \to \infty$ *– "One Bad Apple* Spoils *the Bunch" – we have that*

$$k_j = b_j \frac{1}{c - (ab_{\min})^2}, s_j = b_j \frac{ab_{\min}}{c - (ab_{\min})^2}. \tag{17}$$

Therefore, human capitals and socialization efforts are still proportional to the respective cognitive skills, but the social multiplier reflects the impact of the "one bad apple." It exhibits extreme aversion to the prospect of benefiting from interaction. For a proof, see Appendix A, Proofs.

Proposition 3. *For the CES interactions assumption (16), under* $\frac{1}{\xi} \to -\infty$ *– "The Best*

Individual is the *Role Model" – we have that*

$$k_j = b_j \frac{1}{c - (ab_{\max})^2}, s_j = b_j \frac{ab_{\max}}{c - (ab_{\max})^2}. \tag{18}$$

Human capital investments and socialization efforts are still proportional to the respective cognitive skill, and the social multiplier is now larger than in the previous example of "one bad apple spoils the bunch." Just like Lucas and Moll (2014), the social effect is associated with learning from the "best individual" contacted as the role model. It exhibits extreme optimism to the prospect of benefiting from interactions.

If individuals do not know the cognitive skills of others, but do know their own when they have to make decisions, one could think of the socialization efforts as defining a social portfolio. With this in mind, I develop further the model under the assumption of uncertainty with respect to individuals' cognitive skills for the special cases of "one bad apple spoils the bunch" and of "best individual is the role model" metaphors. The analysis is simplified under the assumption that cognitive shocks are Fréchet-distributed. One could redefine the CES interaction structure so as to highlight social connections instead of human capitals, or even both types of effects, but such an extension is not pursued further here. Many alternative specifications are possible.

2.2.1 Cognitive Shocks

Individual i observes the realization of $b_i = \psi_i$ and then sets (k_i, s_i), which as a consequence do depend on ψ_i. Individuals do not observe the realizations of others' cognitive shocks, and are thus subject to uncertainty about the impact of cognitive shocks of others on their decisions. I examine in further detail the case of "The Best Individual is *the* Role Model" under the assumption that the cognitive shocks are independent and identically Fréchet-distributed random variables.[11] The results are summarized by the following proposition.

Proposition 4. *If individuals set (k_i, s_i) after they have observed their own cognitive shock, $b_i = \psi_i$, and the ψ_i's are independent and identically Fréchet-distributed random variables* *with a cumulative distribution function given by* $\exp\left[-\left(\frac{\psi-m}{\sigma}\right)^{-\chi}\right]$*, where* (m,σ,χ) *are positive parameters, denoting the minimum, scale, and shape parameters, respectively, then* $k_i = \nu\psi_i$ *and* $s_i = a\nu\psi_i\left[m + I^{\frac{1}{I}}\nu\sigma\Gamma\left(1-\frac{1}{\chi}\right)\right]$*, where* ν *is a root of the cubic equation:*

$$\nu = \frac{1}{c - a^2\left[m + I^{\frac{1}{I}}\sigma\Gamma\left(1-\frac{1}{\chi}\right)\right]^2\nu^2}. \tag{19}$$

Depending upon parameter values, this equation may have either one or two feasible solutions, or none.

Feasibility is conceptual similar to condition (15), with $\tilde{a}$ now defined as:

$$\tilde{a} \equiv a\ \left[m + I^{\frac{1}{\chi}}\sigma\Gamma\left(1-\frac{1}{\chi}\right)\right]. \tag{20}$$

The optimal values of the k_i's are again proportional to their respective $b_i = \psi_i$'s and so are the s_i's.

The case of "One Bad Apple Spoils the Bunch" may also be handled by similar techniques that also rely on extreme order statistics but is not pursued further here.[12] "The Best Individual is *the* Role Model" solution is conceptually similar to the benchmark case above (2.1), with an important difference. That is, whereas $\tilde{a}$ in the benchmark case (3) adjusts a on account of the relative dispersion of the b_i's, definition (20) combines the mean of the cognitive shock as a well as its scale parameter, adjusted by a term that accounts for the thickness of the upper tail. The thicker the upper tail of the distribution of the cognitive shocks the smaller is χ, and the smaller is $\Gamma\left(1-\frac{1}{\chi}\right)$, the less the scale parameter contributes to $\tilde{a}$, and the smaller the social multiplier associated with (19). An increase in m, the mean, alternatively in σ, the scale (dispersion) parameter, or a larger χ, a thicker upper tail, increases $\tilde{a}$, narrows the spread between the two feasible roots by increasing the smaller and reducing the larger of the two. This in turn decreases the larger social multiplier and increases the smaller social multiplier.

3 Dynamics

A conventional dynamic analysis of such a model follows from defining an intertemporal objective function for agents, and allowing for the first-order conditions to yield equations exhibiting dynamic adjustment. Let us rewrite the definition of the utility per period[13] (1) as:

$$U_{i,t}(\mathbf{s}_{t-1}; s_{it}; \mathbf{k}_{t-1}, k_{it}) \equiv b_{\tau(i)} k_{it} + a \sum_{j=1, j\neq i}^{I} g_{ij}(\mathbf{s}_{t-1}) k_{it} k_{jt-1} - c\frac{1}{2} k_{it}^2 - \frac{1}{2} s_{it}^2. \tag{21}$$

According to definition (21), it is networking efforts, that is interaction weights at time $t-1$, $\mathbf{s}_{t-1}$, that affect spillovers at time t resulting from actions at time $t-1$. Accordingly, in deciding on her networking efforts and thus interaction weights, agent i anticipates the impact on her utility in the next period. Specifically, agent i seeks to maximize

$$\sum_{t=0}^{\infty} \rho^t U_{i,\tau(i),t}(\mathbf{s}_{t-1}; s_{it}; \mathbf{k}_{t-1}, k_{it}),$$

by choosing sequences of human capital investment and networking efforts $\{\mathbf{k}_{it}\}_0^\infty, \{\mathbf{s}_{it}\}_0^\infty$, taking as given all other agents' contemporaneous decisions $\{\mathbf{k}_{-it}\}_0^\infty, \{\mathbf{s}_{-it}\}_0^\infty$, where $\rho, 0 < \rho < 1$, denotes the discount rate. This optimization problem may be easily modified to allow for depreciation of human capital and of links. The development of the dynamic models that follow depend critically on timing conventions assumed. After much experimentations, the assumptions made allow for a tractable development of the dynamics without sacrificing the potential richness of the interactions between human capital and social networking decisions.

3.1 Joint Evolution of Human Capital and Social Connections

The dynamic analysis is summarized in Proposition 5, which follows next and whose proof is given in the Appendix, Proofs.

Proposition 5. Agents' choices of sequences of human capital investment and networking efforts $\{\mathbf{k}_{it}\}_0^\infty, \{\mathbf{s}_{it}\}_0^\infty$, *taking as given all other agents' contemporaneous decisions* $\{\mathbf{k}_{-it}\}_0^\infty, \{\mathbf{s}_{-it}\}_0^\infty$, $0 < \rho < 1$, *satisfy:*

Part A. the system of difference equations

$$\mathbf{k}_t = \frac{1}{c}\mathbf{b} + \frac{a}{c}\mathbf{G}(\mathbf{s}_{t-1})\mathbf{k}_{t-1}; \tag{22}$$

$$\mathbf{s}_t = a\rho[diag\ \mathbf{k}_{t+1}]\frac{\partial \mathbf{G}(\mathbf{s}_t)}{\partial \mathbf{s}_t}\mathbf{k}_t, \tag{23}$$

where $[diag\ \mathbf{k}_{t+1}]$ *denotes an* $I \times I$ *matrix with the elements of* $\mathbf{k}_{t+1}$ *along the main diagonal,* $g_{ij}(\mathbf{s}_t)$ *is as defined by (2), and* $\frac{\partial \mathbf{G}(\mathbf{s}_t)}{\partial s_t}$ *denotes a matrix with the terms* $\frac{\partial \mathbf{g}_{ij}(s_{it})}{\partial s_t}$ *as its* i*th row.*

Part B. The steady state values of the system (22–23) (k_i^*, s_i^*) *coincide with those of the static case (10–11), provided that one adjusts for the fact that to* a *in (11) there corresponds* $a\rho$ *in (23).*

Part C. The deviations $\Delta s_{it} = s_{it} - s_i^*$, $\Delta k_{it} = k_{it} - k_i^*$, *defined as vectors,* $\Delta \mathbf{k}_t, \Delta \mathbf{s}_t$ *satisfy the dynamical system*

$$\Delta \mathbf{k}_{t+1} = \frac{a}{c}\mathbf{G}(\mathbf{s}^*)\Delta \mathbf{k}_t, \tag{24}$$

$$\Delta \mathbf{s}_t = \frac{a\tilde{a}(\mathbf{b})\rho\vartheta}{c}\mathbf{G}(\mathbf{s}^*)\Delta \mathbf{k}_t. \tag{25}$$

The system is locally dynamically stable for both non-zero steady state values of $(\mathbf{k}^*, \mathbf{s}^*)$, *defined by Proposition 1.*

Some remarks are in order. The derivation of the first-order conditions for k_{it}'s ignores, in the sense of Nash equilibrium, the effect that agent i's setting of k_{it} has on the spillovers to all agents in period t, taking them as given.

I note that the system is locally stable near both non-zero steady states, as in the case of Cabrales *et al.* (2011). This result poses issues of equilibrium selection in the underlying multi-person game, which are not pursued further in this paper. Further below, I show that the basic dynamic model here also underlies models which allow for individuals to make intergenerational transfers to their children. I note that stability of the human capital process implies that of the networking efforts as well provided that $\rho < \varpi_{-1} < 1$.

3.2 Evolution of Human Capital with Exogenous Social Connections

An examination of the evolution of human capital, given social connections, offers interesting contrast for the case of endogenous connections. Assuming that $\{\mathbf{s}_t\}_{t=1}^{\infty}$ is exogenous and given, and taking $\{\mathbf{k}_{-i,t}\}_{t=1}^{\infty}$, as given, individual i chooses $\{\mathbf{k}_{i,t}\}_{t=1}^{\infty}$, so as to maximize lifetime utility according to (21). Under the assumptions of Nash equilibrium, human capitals satisfy the sequence of difference equations (22), now rewritten in matrix form as:

$$\mathbf{k}_t = \frac{1}{c}\mathbf{b}_t + \frac{a}{c}\mathbf{G}_{-diag}(\mathbf{s}_{t-1})\mathbf{k}_{t-1}. \tag{26}$$

To see the properties of this process, let us assume that both $\mathbf{s}_t$ and $\mathbf{b}_t$ are constant, $\mathbf{s}, \mathbf{b}$. Then, (26) admits a steady state, given by:

$$\left[\mathbf{I} - \frac{a}{c}\mathbf{G}(\mathbf{s})\right] c\mathbf{k} + a\mathrm{diag}\mathbf{G}(\mathbf{s})\mathbf{k} = \mathbf{b}. \tag{27}$$

As we argued above, for a large number of agents, the diagonal elements vanish, and the second term on the lhs of (26) is approximately equal to zero.

The special properties of $\mathbf{G}(\mathbf{s})$ allow deriving conditions under which $\left[\mathbf{I} - \frac{a}{c}\mathbf{G}(\mathbf{s})\right]^{-1}$ exists. Specifically, since $\mathbf{G}(\mathbf{s})$ is symmetric and positive, all of its eigenvalues are real. It has a maximal simple eigenvalue, r, which is positive, and larger from the absolute values of all its other eigenvalues. Then, by Theorem III, Debreu and Herstein (1953), $\left[\mathbf{I} - \frac{a}{c}\mathbf{G}(\mathbf{s})\right]^{-1}$ exists is positive, if and only if

$$\frac{1}{r} > \frac{a}{c}. \tag{28}$$

As Cabrales *et al.* (2011), show, the maximal eigenvalue is given by $\frac{\overline{x^2}(\mathbf{s})}{\overline{x}(\mathbf{s})}$ and corresponds to $\mathbf{s}$ as an eigenvector. Furthermore, by Lemma 3, Cabrales et al. (2011), p. 353,

$$\left[\mathbf{I} - \frac{a}{c}\mathbf{G}(\mathbf{s})\right]^{-1} = \mathbf{I} + \frac{a}{c}\frac{1}{1 - \frac{a}{c}\frac{\overline{x^2}(\mathbf{s})}{\overline{x}(\mathbf{s})}}\mathbf{G}(\mathbf{s}).$$

Thus, condition (28) that the maximal eigenvalue must satisfy suffices for the positivity of $\overline{x}(\mathbf{s}) - \frac{a}{c}\overline{x^2}(\mathbf{s}^2)$, and thus of the second term of the expression for the inverse above. The steady state value for $\mathbf{k}^*$ becomes:

$$\mathbf{k}^* = \frac{1}{c}\mathbf{b} + \frac{a}{c}\frac{1}{1 - \frac{a}{c}\frac{\overline{x^2}(\mathbf{s})}{\overline{x}(\mathbf{s})}}\mathbf{G}(\mathbf{s})\frac{1}{c}\mathbf{b}. \tag{29}$$

For the linear dynamical system (26), the unique steady state is stable, provided its maximal eigenvalue is less than 1, which is equivalent with condition (28).

Human capitals at the steady state, given by (29), consist of two terms of which the second only reflects the effects of social interactions. Inspection of the second term in the rhs of (29) suggests that it consists of a vector whose term i is

$$\frac{a}{c}\,\frac{s_i}{\sum_i s_i}\,\frac{1}{c}\frac{1}{1-\frac{a}{c}\frac{\overline{x^2}(\mathbf{s})}{\overline{x}(\mathbf{s})}}\,\mathbf{s}\cdot\mathbf{b}. \tag{30}$$

It follows from (29) and (30), that human capitals consist of two terms: one is the autarkic value, $\frac{1}{c}b_i$; the second, above, involves a term that is common to all that is weighted by the an individual's socialization effort, relative to the sum of all efforts. Clearly, when the social connections are not optimized, the exogenously social connections do matter. Both the original dispersion of cognitive skills and of the social connections contribute to the dispersion of human capital across the population. In contrast, it is optimizing over social connections that renders human capitals and social connections proportional to the b's. Finally, without optimization over social networking, Eq. (23) are not part of the first-order conditions, and no equilibrium multiplicity arises. Given social connections, human capitals are uniquely defined on the transition to and at the steady state.

Clearly, when the social connections are not optimized, the exogenously social connections do matter. Both the original dispersion of cognitive skills and of the social connections contribute to the dispersion of human capital across the population. Allowing for heterogeneity in the parameter a, which we interpret as proxying for social competence (in the terminology of Clark (2014)) or for non-cognitive skills, or for its stochastic dispersion across the population, which we explore in section 5.3 further below, adds an additional exogenous source of dispersion in the evolution of human capitals.

3.2.1 A Stochastic Extension and the Upper Tail of the Distribution of Human Capitals

By taking the evolution of human capital in relation to social connections as given, that is by assuming that Eq. (26) holds as an ad hoc rule, we allow for stochastic shocks to cognitive as well as non-cognitive skills. We recall the specification of cognitive shocks in section 2.2.1 above and assume that the (column) vectors $\mathbf{\Psi}_t = (\psi_{1,t}, \ldots, \psi_{I,t})$ are defined to represent the full cognitive effect, where $\psi_{i,t} = \frac{1}{c} b_{i,t}$, with $\mathbf{\Psi}_t$ being a random vector that is independently and identically distributed over time. That is, the sequence of $\{\mathbf{\Psi}_0, \ldots, \mathbf{\Psi}_t\}$ is assumed to be a stationary vector stochastic process. In addition, we assume that social connections are exogenous but random. That is, the social networking efforts are denoted by $\mathbf{\Phi}_t = (\phi_{1,t}, \ldots, \phi_{I,t})$, so that instead of (26) we now have:

$$\tilde{\mathbf{k}}_t = \mathbf{\Psi}_t + \tilde{\mathbf{G}}(\mathbf{\Phi}_t)\tilde{\mathbf{k}}_{t-1}, t = 1, \ldots, \tag{31}$$

with a given $\tilde{\mathbf{k}}_0$. For the purpose of analytical convenience and without loss of generality, we assume that the social interactions matrix $\tilde{\mathbf{G}}_t = \tilde{\mathbf{G}}(\mathbf{\Phi}_t)$ is defined to include the diagonal terms too. Proposition 6 establishes a limit result fore the upper tail of the distribution of human capitals. For the details of the proof, see Appendix, Proofs. The result is obtained by adapting Theorems A and B, Kesten (1973), as discussed in more detail in the Appendix.

Proposition 6. Let the pairs $\left\{\tilde{\mathbf{G}}_t, \mathbf{\Psi}_t\right\}$ *be independently and identically distributed elements of a stationary stochastic process with positive entries, where* $\tilde{\mathbf{G}}_t$ *are* $I \times I$ *matrices and* $\mathbf{\Psi}_t$ *are* $I-$ *vectors. Under the additional conditions of Theorems A and B, Kesten (1973; 1974) and the assumption of the function* $||m|| = \max_{|y|=1} |ym|$, *where* y *denotes an* I *row vector, and* m *denotes an* $I \times I$ *matrix, as the matrix norm* $||\cdot||$ *for* $I \times I$ *matrices, and* $|\cdot|$ *denotes the Euclidian norm, then:*

Part A. The series

$$\mathbf{K} \equiv \sum_{t=1}^{\infty} \tilde{\mathbf{G}}(\mathbf{\Phi}_1) \cdots \tilde{\mathbf{G}}(\mathbf{\Phi}_{t-1})\Psi_t \tag{32}$$

converges w. p. 1, and the distribution of the solution $\tilde{\mathbf{k}}_t$ *of (31) converges to that of* $\mathbf{K}$, *independently of* $\tilde{\mathbf{k}}_0$.

Part B. For all elements x *on the unit sphere in* $I\!R^I$, *under certain conditions, there exists a positive constant* κ_1, *and*

$$\lim_{v\to\infty} v^{\kappa_1} Prob\{x\mathbf{K} \geq v\} \tag{33}$$

exists, is finite and for all elements x *on the unit sphere of* $I\!R^I$, *and for all the elements on the positive orthant of the unit sphere is strictly positive.*

Proposition 6, Part A merely establishes properties of the limit of the vector of human capitals. Part B relies on these properties to establish a Pareto (power) law for the upper tails of the joint distribution of human capitals, characterized by (33). Its significance lies in that a power law is obtained for a sequence of random vectors, not just a scalar random variable. Its intuition is straightforward.[14] Given a non-trivial initial value for the cognitive shocks, $\Psi(1)$, and an arbitrary initial value for human capitals, $\tilde{\mathbf{k}}_0$, the dynamic evolution of human capital according to (31) keeps positive the realizations of human capital, while the impact of spillovers is having an overall contracting effect that pushes the realizations and thus the distributions of human capital, too, towards 0. The distribution is prevented from collapsing at 0 by the properties of the contemporaneous cognitive shocks, Ψ_t, and from drifting to infinity by the contracting effect of the spillovers. The contracting effect results from the combination of two key requirements: First, a condition, condition (98) in the Appendix, which requires that there exists a positive constant κ_0, for which the expectation of the minimum row sum of the social interactions matrix raised to the power of κ_0, grows with the number of agents I faster than $\sqrt{I}$, roughly speaking; and second, the geometric mean of the limit of the sequence of norms of the social interactions matrix is positive but less than 1.[15]

Thus, the upper tail of the joint distribution of $x\mathbf{K}$, for all elements on the unit sphere of $\mathbb{R}^I$, is thickened by the combined effect of the contracting spillovers and tends to a power law, $\propto v^{-\kappa_1}$, with an exponent κ_1 which is constant. This result is sufficient for the distribution of human capital in the entire economy to also have a Pareto upper tail. Let f_{k_i} denote the limit distribution of $k_i,\ i = 1, \ldots, I$. Then, the economy-wide distribution of human capitals is given by $\sum_i \#\{i\} f_{k_i}(k)$, where $\#\{i\}$ denotes the relative proportion of types i agents. Following Jones (2014), one may approximate the value of the Pareto exponent κ_1 in terms of the parameters of the distribution of $\tilde{\mathbf{G}}, \Psi_t$.

The scalar counterpart of the conditions of Kesten's theorems have been extensively invoked in the economics literature. E.g., see Gabaix (1999), 761–762, whose approach can be the starting point for linking the magnitude of the Pareto exponent approximating the upper tail to the parameters of the underlying distribution of interest.

4 Some Consequences for Inequality

Sticking to an interpretation of actions as human capital investments, the variation across individual types, as expressed in the b_i's, can then be seen as a primitive determinant of the distribution of human capital across a population, that is about "what you know." Here, we see that optimized individual human capitals are proportional to the individuals' cognitive skills, with the factors of proportionality being functions of the distribution of the b_i's across individuals. This can be demonstrated to hold for many different interaction structures. Consequently, individuals' utilities do depend on the distribution of the b_i's across individuals in more complicated ways. In the simplest formulation, they depend on the first and second moments of the distribution of the b_i's across types only. Individualizing the interactions structure by including functions of the b_i's lead to more complicated moments of the b_i's. Fully individualizing the interaction weights, or allowing for homogeneity of degree less than, or greater than, one do not change the basic conclusion, namely that the outcomes are proportional to b_i's, albeit with different multipliers.

In view of the optimal solution above for either the static or the steady state one in the dynamic case, we may compute the corresponding optimum value of the individuals' utility functions. By using (13) and (14), the value becomes:

$$U_{i,\tau(i)}(\mathbf{s}^*, \mathbf{k}^*) = \frac{1}{2}\vartheta b_{\tau(i)}^2. \tag{34}$$

In the case of autarky, $U_{i,\tau(i),\mathrm{aut}} = \frac{1}{2}\frac{1}{c}b_{\tau(i)}^2$. Since from (14), if ϑ exists, which is ensured by the condition that (13)–(14) have at least one solution, then $\vartheta > \frac{1}{c}$.

Thus individuals' self-organizing into a social network Pareto-dominates autarky, and the optimum values of the quantity ϑ summarizes the impact of social networking, which includes the consequences of the human capital decisions that that makes possible, on an individual's welfare. This is true for either of the two sets of values of (ϖ, ϑ), the two sets of roots of (13)–(14), defined by Proposition 1. However, greater dispersion of the b_i's, that is a larger value of $\tilde{a}$, is associated with a smaller spread between the two alternative solutions. The greater is $\tilde{a}$, the greater is the smaller of the two roots, $\vartheta_{\min}$, and the smaller the larger of the two, $\vartheta_{\max}$. Holding a constant, this occurs if $\frac{\overline{x^2}(\mathbf{b})}{\overline{x}(\mathbf{b})}$ is greater. The feasibility condition

(15) provides an upward bound on $\tilde{a}$. Greater dispersion of the b_i's, as indicated by a larger value of $\tilde{a}$, narrows the advantages, as expressed by the welfare value of outcomes, associated with the larger social multiplier, relative to the smaller one. Too much dispersion in cognitive skills renders socially advantageous networking infeasible.

Therefore, an attractive interpretation of this result is that the equilibrium solution for ϑ, from Proposition 1, Part A, summarizes individuals' benefits from self-organization into social networks. Below, I take up the question about how the option to optimize the social interactions weights affects outcomes about human capital at the steady state, that is, "how whom you know" affects intergenerational transfers. In the models examined above, at the steady state, all outcomes are proportional to the respective b's, when social connections are optimized. So, the variation of optimal actions and optimum utility across individuals separates naturally into the impact of networking opportunities and of cognitive skill, being proportional, to the b_i, respectively b_i^2, with ϑ, the factor of proportionality, reflecting the effect of the entire distribution of the b_i's via social networking. Thus, in a model where proxies for cognitive skills are inherited, this feature may be relied on, in a model with a finite number of overlapping generations, or in infinite-horizon model, to express inheritability. The question then becomes to what extent "the human wealth you inherit" influences "whom you know."

4.1 Unstable Social Structures

When social connections are exogenous, a great number of possibilities arises. The development in section 3.2 shows that the stability of the dynamic evolution of human capital depends on the properties of the social network, relative to the parameters of the utility function. Thus, when the social network does not satisfy conditions for stability, that is when

$$\frac{\overline{x}(\mathbf{s})}{\overline{x^2}(\mathbf{s})} < \frac{a}{c}, \tag{35}$$

and depending on initial conditions, one may think of whether it might be possible to have sets of socially networked individuals whose human capitals converge over time, and while for others they diverge. Given any given set of social networking efforts, it is straightforward to obtain conditions under which such groupings of individuals are feasible. Specifically, it is straightforward to show that given that there is a grouping of $h-1$ individuals for whom

$$\frac{\overline{x}_{h-1}(\mathbf{s})}{\overline{x^2}_{h-1}(\mathbf{s})} < \frac{a}{c}, \tag{36}$$

then the lhs of (36) above increases, that is, $\frac{\overline{x}_{h-1}(\mathbf{s})}{\overline{x^2}_{h-1}(\mathbf{s})} < \frac{\overline{x}_{h}(\mathbf{s})}{\overline{x^2}_{h}(\mathbf{s})}$, provided that individual h being added satisfies: $s_i > \frac{\overline{x^2}_{h-1}(\mathbf{s})}{\overline{x}_{h-1}(\mathbf{s})}$. That is, a prospective new member of the group must have sufficiently high networking effort in order to improve social networking for the entire group she stands to join. Thus, by successive addition of such individuals the inequality sign in the infeasibility condition (36) would be reversed and the condition for stability established. Recall that the spirit of the model is that there exist many different individuals of each type. Therefore, this ought to be understood as how different types of individuals with given social networking efforts may self-organize into different social networks.

Applying these models to dynamic settings, where one may compare between given weights, perhaps representing a given social structure, and optimized weights, one may thus distinguish between given relationships, like familial ones, versus social networking across familial relationships.

5 Overlapping Generations linked through Intergenerational Transfers

I consider next dynamic versions of the model that allows for intergenerational transfers. I consider first transfers of wealth, whereby individuals start their lives with a given level of wealth in the form of human capital, denoted by $k_{i,t}$, which they receive from their parents. They give birth to a child, to whom they transfer wealth equal to $k_{i,t+1}$. We let the utility function, given in (1), $U_{i,t}(\mathbf{s}_t, \mathbf{k_t})$, denote the period t payoff for individual i, let dynastic utility be identified with the value function associated with the dynamic process for each dynasty be denoted by $\mathcal{U}_{i,t}(k_{i,t})$. That is, dynastic utility following the canonical typology of Barro (1974) is defined in the standard fashion for dynamic programming problems via the maximization of remaining utility:

$$\mathcal{U}_{i,t}(k_{i,t}) = \max_{s_{i,t},k_{i,t+1}} : \left\{U_{i,t}(\mathbf{s}_t, \mathbf{k_t}) + \rho \mathcal{U}_{i,t}(k_{i,t+1})\right\}, \tag{37}$$

where utility per period, $U_{i,t}(\mathbf{s}_t, \mathbf{k_{i,t}})$, is given by (1):

$$U_{i,\tau(i)}(\mathbf{s}, \mathbf{k}) \equiv b_{\tau(i)} k_{i,t} + a \sum_{j=1, j\neq i}^{I} g_{ij}(\mathbf{s}_t) k_{i,t} k_{j,t} - c\frac{1}{2}k_{i,t}^2 - \frac{1}{2}s_{i,t}^2 - k_{i,t+1}.$$

In this formulation each parent at t decides on a transfer to her child, $k_{i,t+1}$, and on the networking effort, $s_{i,t}$, that she avails herself from, given the transfer which she herself received from her own parent, $k_{i,t}$, so as to maximize her lifetime utility. Note that whereas the parent incurs the resource cost, $k_{i,t+1}$, of the transfer to the child, the child incurs the adjustment cost, that is the quantity $\frac{1}{2}k_{i,t}^2$ for the individuals who are the parents at t. Dynastic utility is defined as the sum of her own period t utility plus the discounted sum of the maximum utilities of her descendants. It is perfectly feasible to develop this model, but I note that by making the transfer to the child and her own networking efforts as simultaneous decisions, the child does not benefit from the parent's networking. In such a model, there is no human capital accumulation, since each individual lives for one period, nor growth (although exogenous growth to the productivity of human capital could be introduced). This otherwise standard model exhibits the property of the life cycle theory, in its being isomorphic to a model of a single decision maker who maximizes an infinite sum of utilities with respect to a sequence of decisions, $\{k_{i,t+1}, s_{i,t}\}_{t=0}^{\infty}$. I do not pursue this model further.

5.1 Intergenerational Transfers of Wealth and of Social Connections

A richer model and analytically more tractable one may be obtained if we assume that individuals have finite lifetimes and are present in the economy in overlapping generations. I start with two overlapping generations, but do note however that a minimum of three overlapping generations will be necessary to express Heckman's concern about allowing for at least two periods of investment in a child's cognitive and non-cognitive skills. That is, it is critical [see Cunha and Heckman (2007) and Cunha, Heckman and Schennach (2010)] for the acquisition of cognitive and non-cognitive skills to interact — there is dynamic complementarity among them — and investments in certain ages are more critical then in other ages. Moreover, these come earlier for cognitive capabilities, later for non-cognitive capabilities, and vary depending on the particular biological capability. Three-overlapping generations is the minimum number that allows for direct effects between grandparents and grandchildren. Heckman and Mosso (2014) emphasize, however, there have to be at least four periods in individuals' lifetimes, with two periods for a passive child who makes no economic decisions but who benefits from parental investment in the form of goods, and two periods as a parent. This requires, of course, going beyond the standard two-overlapping generations models used by many life cycle models. See section 5.4 below for steps in this direction.

The fact that parents are assumed to coexist with their children naturally allows me to model that children may avail themselves of the social connections of their parents. Such a natural "transfer in kind" can coexist with a wealth transfer. Both types of transfers are central features of the models that follow.

5.1.1 A Two-Overlapping Generations Model of Intergenerational Transfers

Let subscripts y, o refer to individuals when they are *y*oung, *o*ld, respectively, and let time subscripts refer to when the respective quantity is operative. A member of the generation born at t receives a transfer $k_{y,i,t}$ from her parent when young; she herself takes advantage at time t of social connections chosen by her parent's generation: $\mathbf{s}_{y,t-1}$. Her cognitive skills are given: $b_{y,i,t}, b_{o,i,t+1}$. She chooses human capital investment and networking effort $(k_{o,i,t+1}, s_{y,i,t})$; she benefits in period $t+1$ from $k_{o,i,t+1}$; she and her entire generation benefit from $\mathbf{s}_{y,t}$ in time $t+1$. She chooses an endowment to her child in the form of human capital, $k_{y,i,t+1}$, and networking effort, $s_{o,i,t+1}$, from which her child benefits in the first period of her own life at time $t+1$. We assume that the resource cost of investment $k_{o,i,t+1}$ is incurred in period t, but the adjustment costs is incurred in $t+1$ (when the benefits are also realized); consistently, the resource cost of $k_{y,i,t+1}$ is incurred in period $t+1$, but the parent anticipates that the adjustment costs are incurred by the child in $t+1$.

In the remainder of this section we generalize the static model introduced above and obtain a system of dynamic equations. It coincides with that system in the special case of cognitive skills which are equal across young and old and invariant over time: $b_i = b_{y,i,t} = b_{o,i,t+1}$.

It is important to clarify the relevant peer groups underlying this formulation. With two overlapping generations, we may define the peer groups for young generation t at time t as the members of generation who were born at $t-1$ when they are old at time t. That is, the members of generation t benefit in period t from the human capitals $\mathbf{k}_{o,t}$ and the social networking of their parents' generation, $\mathbf{s}_{o,t}$. When they are old in period $t+1$ they benefit by the human capitals and social contacts the members of their own generation themselves decided on, $\mathbf{k}_{y,t}, \mathbf{s}_{y,t}$. In other words, in their first-period decisions about social connections, individuals are conscious of the fact that they themselves would benefit from their own social connections when they are old; in their second-period decisions about social connections, they are conscious of the fact that their children would benefit from their own second-period social connections when their children are young. Therefore, all second-period decisions are in effect intergenerational transfers of capital and social connections. In the absence of uncertainty, all decisions are of course made simultaneously, but being explicit about "timing" of networking efforts would be crucial with sequential resolution of uncertainty, when such uncertainty is introduced, as in section 5.2 below.

That is, the decision problem for a member of generation t, born at time t, is to choose $\{k_{o,i,t+1}, k_{y,i,t+1}; s_{y,i,t,}, s_{o,i,t+1}\}$, given $\{k_{y,i,t}, \mathbf{s}_{o,t}\}$. We obtain first-order conditions for each generation's decision variables by first defining the value functions and using the envelope property. The results are summarized in the proposition that follows; the proof is in the Appendix, Proofs.

Proposition 7.

Part A. The value functions for individual i as of time t and for her child as of time $t+1$ are defined respectively as follows, $\mathcal{V}_i^{[t]}(k_{y,i,t}, \mathbf{s}_{o,t}), \mathcal{V}_i^{[t+1]}(k_{y,i,t+1}, \mathbf{s}_{o,t+1})$, associated with an *individual's lifetime utility when he is young at t and when he is old at $t+1$, we have:*

$$\mathcal{V}^{[t]}(k_{y,i,t}, \mathbf{s}_{o,t})$$

$$= \max_{\{k_{o,i,t+1}, k_{y,i,t+1}; s_{y,i,t,}, s_{o,i,t+1}\}} \left\{ b_{y,i,t}k_{y,i,t} + a\sum_{j\neq i} g_{ij}(\mathbf{s}_{o,t})k_{y,i,t}k_{o,j,t} - \frac{1}{2}ck^2_{y,i,t} - \frac{1}{2}s^2_{y,i,t} - k_{o,i,t+1} \right.$$

$$\left. +\rho\left[b_{o,i,t+1}k_{o,i,t+1} + a\sum_{j\neq i} g_{ij}(\mathbf{s}_{y,t})k_{o,i,t+1}k_{y,j,t} - \frac{1}{2}ck^2_{o,i,t+1} - \frac{1}{2}s^2_{o,i,t+1} - k_{y,i,t+1}\right] + \rho\mathcal{V}_i^{[t+1]}(k_{y,i,t+1}, \mathbf{s}_{o,t+1})\right\};$$

$$\mathcal{V}_i^{[t+1]}(k_{y,i,t+1}, \mathbf{s}_{o,t+1})$$

$$= \max_{\{k_{o,i,t+2}, k_{y,i,t+2}; s_{y,i,t+1,}, s_{o,i,t+2}\}} \left\{ b_{y,i,t+1}k_{y,i,t+1} + a\sum_{j\neq i} g_{ij}(\mathbf{s}_{o,t+1})k_{y,i,t+1}k_{o,j,t+1} - \frac{1}{2}ck^2_{y,i,t+1} - \frac{1}{2}s^2_{y,i,t+1} - k_{o,i,t+2} \right.$$

$$\left. +\rho\left[b_{o,i,t+2}k_{o,i,t+2} + a\sum_{j\neq i} g_{ij}(\mathbf{s}_{y,t+1})k_{o,i,t+2}k_{y,j,t+1} - \frac{1}{2}ck^2_{o,i,t+2} - \frac{1}{2}s^2_{o,i,t+2} - k_{y,i,t+2}\right] + \rho\mathcal{V}_i^{[t+2]}(k_{y,i,t+2}, \mathbf{s}_{o,t+2})\right\}$$

Part B. The first-order conditions with respect to $(k_{y,i,t+1}, k_{o,i,t+1})$ in vector form yield:

$$\mathbf{k}_{y,t+1} = \frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_{y,t})\mathbf{G}(\mathbf{s}_{o,t+1})\mathbf{k}_{y,t} + \frac{1}{c}\mathbf{b}_{y,t+1} + \frac{a}{c^2}\mathbf{G}(\mathbf{s}_{o,t+1})\mathbf{b}_{o,t+1} - \frac{1}{c\rho}\left[\mathbf{I} + \frac{a}{c}\mathbf{G}(\mathbf{s}_{o,t+1})\right]\mathbf{1}. \quad (38)$$

$$\mathbf{k}_{o,t+1} = \frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_{o,t})\mathbf{G}(\mathbf{s}_{y,t})\mathbf{k}_{o,t} + \frac{1}{c}\mathbf{b}_{o,t+1} + \frac{a}{c^2}\mathbf{G}(\mathbf{s}_{y,t})\mathbf{b}_{y,t} - \frac{1}{c\rho}\left[\mathbf{I} + \frac{a}{c}\mathbf{G}(\mathbf{s}_{y,t})\right]\mathbf{1}. \quad (39)$$

These are all positive provided that $\mathbf{b}_{o,t+1} - \frac{1}{\rho}\mathbf{1} > 0, \mathbf{b}_{y,t+1} - \frac{1}{\rho}\mathbf{1} > 0$.

The first order conditions with respect to $(s_{y,i,t,}, s_{o,i,t+1})$ are:

$$s_{y,i,t} = \rho a k_{o,i,t+1}\sum_{j=1,j\neq i}^{I}\frac{\partial g_{ij}}{\partial s_{y,i,t}}(\mathbf{s}_{y,t})k_{y,j,t}; \quad (40)$$

$$s_{o,i,t+1} = \rho a k_{y,i,t+1}\sum_{j=1,j\neq i}^{I}\frac{\partial g_{ij}}{\partial s_{o,i,t+1}}(\mathbf{s}_{o,t+1})k_{o,j,t+1}; \quad (41)$$

Part C. Sufficient conditions for the invertibility of $\mathbf{I} - \frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_o)\mathbf{G}(\mathbf{s}_y)$ and thus for the existence of meaningful steady state values of (38–39) is that the product of $\left(\frac{a}{c}\right)^2$ and of the largest eigenvalues of each of the positive matrices $\mathbf{G}(\mathbf{s}_o), \mathbf{G}(\mathbf{s}_y)$ be less than 1:

$$\left(\frac{a}{c}\right)^2 \cdot \frac{\overline{x^2}(\mathbf{s}_o)}{\overline{x}(\mathbf{s}_o)} \cdot \frac{\overline{x^2}(\mathbf{s}_y)}{\overline{x}(\mathbf{s}_y)} < 1. \quad (42)$$

Part D. The steady state solutions of (38–39) may be written out in closed form because

$$\left[\mathbf{I}-\left(\frac{a}{c}\right)^2\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)\right]^{-1}=\mathbf{I}+\frac{\left(\frac{a}{c}\right)^2\cdot\frac{\overline{x^2}(\mathbf{s}_o)}{\overline{x}(\mathbf{s}_o)}\cdot\frac{\overline{x^2}(\mathbf{s}_y)}{\overline{x}(\mathbf{s}_y)}}{1-\left(\frac{a}{c}\right)^2\frac{\overline{x^2}(\mathbf{s}_o)}{\overline{x}(\mathbf{s}_o)}\cdot\frac{\overline{x^2}(\mathbf{s}_y)}{\overline{x}(\mathbf{s}_y)}}\mathbf{G}(\mathbf{s}_y;\mathbf{s}_o), \tag{43}$$

where the matrix $\mathbf{G}(\mathbf{s}_y;\mathbf{s}_o)$ *is defined via its* i,j *element as:*

$$\mathbf{G}(\mathbf{s}_y;\mathbf{s}_o)_{ij}=\frac{\sum_\ell s_{y,\ell}s_{o,\ell}}{I\overline{x}(\mathbf{s}_y)I\overline{x}(\mathbf{s}_o)}\ s_{y,i}s_{o,j}. \tag{44}$$

The system of linear difference equations (38–39) is uncoupled in $(\mathbf{k}_{y,t},\mathbf{k}_{o,t})$, given $(\mathbf{s}_{y,t},\mathbf{s}_{o,t},\mathbf{s}_{o,t+1})$. Their steady state solutions are thus easily characterized, in terms of the inverse of $\mathbf{I}-\frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_o)\mathbf{G}(\mathbf{s}_y)$. Since the largest eigenvalue of $\mathbf{G}(\mathbf{s}_o)\mathbf{G}(\mathbf{s}_y)$ is bounded upwards by the product of the largest eigenvalues of $\mathbf{G}(\mathbf{s}_o)$ and $\mathbf{G}(\mathbf{s}_y)$ [Debreu and Herrstein (1953); Merikoski and Kumar (2006), Thm. 7, 154–155], the inverse exists, provided that the product of $\frac{a^2}{c^2}$ with the largest eigenvalues of $\mathbf{G}(\mathbf{s}_o)$ and of $\mathbf{G}(\mathbf{s}_y)$ is less than 1. The characterization of the steady state solution in full detail in section 5.1.4 below allows us to examine these sufficient conditions further.

In the case of three-overlapping generations, that is when children coexist with their parents and their grandparents, we will have an additional set of equations for the respective magnitudes associated with $y$outh, $a$dulthood and $o$ld-age, $(k_{y,i,t},k_{a,i,t+1},k_{o,i,t+2};s_{y,i,t},s_{a,i,t+1},s_{o,i,t+2})$. An individual born at t, will take as given $(k_{y,i,t},s_{y,i,t})$ and choose $(k_{a,i,t+1},k_{o,i,t+2},k_{y,i,t+3};s_{a,i,t+1},s_{o,i,t+2},s_{y,i,t-}$ Intuitively, one would expect that the additional first-order conditions would introduce additional multiplicative terms to the matrix defining the dynamical system and additional terms multiplying the respective cognitive skills vectors. That is, the endowment of cognitive skills in each period of the life cycle introduce life cycle effects into the model, being weighted by the respective social interactions matrix, as in $\frac{1}{c}\frac{a}{c}\mathbf{G}(\mathbf{s}_{y,t})\mathbf{b}_{y,t}$ in Eq. (39) above. Given the pattern of recurrence, we can guess what the counterpart of (39) should look like. Since the respective endowments are not equal across time, steady state values for human capitals differ at different stages of the life cycle.

It is known from research on models with more than two overlapping generations [see Azariadis *et al.* (2004) and references there in] that more than two overlapping generations models usher in considerably more complicated properties in general equilibrium contexts.[16] It is therefore interesting that complicating the demographic structure of the model leaves tractable the structure that determines the dynamics of the model. Working through the derivations formally in order to derive the counterpart of (39) confirms, in fact, this intuition.

5.1.2 Social Effects in Intergenerational Wealth Transfer Elasticities

Interpreting human capital $k_{y,i,t}$ as initial wealth for a member of the generation born at t allows us to compute intergenerational wealth elasticities. This allows for a deeper understanding of estimated intergenerational wealth transfer elasticities.

We work from (38) and define the elasticity of $k_{y,i,t+1}$ with respect to $k_{y,i,t}$ and account only for direct effects, $\mathrm{EL}_{k_{y,i,t}}^{k_{y,i,t+1}} = \frac{\partial k_{y,i,t+1}}{\partial k_{y,i,t}} \frac{k_{y,i,t}}{k_{y,i,t+1}}$, that is, effects on i's decisions as opposed to the impact of i's decisions on decisions of other agents, which feed back to agent i's decisions. We write it for brevity as $\mathrm{EL}(k)_t^{t+1}$. It is easiest to see the effect under the assumption that social networking is given. Then, from (38) and (39) we have a direct effect, $\frac{\partial k_{y,t+1}}{\partial k_{y,i,t}} = \frac{a^2}{c^2} \left[\mathbf{G}(\mathbf{s}_{y,t})\mathbf{G}(\mathbf{s}_{o,t+1})\right]_{\text{col } i}$. This effect is simply the increase in the transfer to the child, $k_{y,i,t+1}$, from an increase in first period wealth received by a member of the tth generation. This is determined from trading off the resource cost of the transfer, which is incurred by the parent in period $t+1$, with the utility increase the parent enjoys from the benefit to the child when the transfer is received in period $t+1$. This is why both adjacency matrices, $\mathbf{G}(\mathbf{s}_{y,t})$ and $\mathbf{G}(\mathbf{s}_{o,t+1})$, are involved in the expression for $\frac{\partial k_{y,i,t+1}}{\partial k_{y,i,t}}$.

However, because the transfer to the child, $k_{y,i,t+1}$, and the parent's social networking effort when old, $s_{o,i,t+1}$, are jointly determined, the full benefit to the child also reflects how the change in the parent's social networking effort influences the human capital spillovers, which are associated with the parents' human capitals in period $t+1$, the second period of their lives. We see from (39) that $k_{o,i,t+1}$ is determined, given $(k_{o,i,t}, s_{y,i,t}, s_{o,i,t})$. Thus, in using the interdependence of $(k_{y,i,t+1}, s_{o,i,t+1})$, as in (41), to express the effect of $k_{y,i,t}$ on $k_{y,i,t+1}$ via $s_{o,i,t+1}$, we have:

$$\frac{\partial s_{o,i,t+1}}{\partial k_{y,i,t+1}} = \rho a \sum_{j=1, j\neq i}^{I} \frac{\partial g_{ij}}{\partial s_{o,i,t+1}}(\mathbf{s}_{o,t+1}) k_{o,j,t+1}, \tag{45}$$

given $\mathbf{k}_{o,t+1}, \mathbf{s}_{o,-i,t+1}$. Therefore, an effect is generated on $k_{y,i,t+1}$ due to its dependence on $s_{o,i,t+1}$, which is obtained by partially differentiating the rhs of (38) with respect to $s_{o,i,t+1}$.

This analysis comes in handy when we examine the impact of differences in the parent's or in the child's own cognitive skills on the transfer to the child. From (38) applied for time t we have that an individual with higher first-period cognitive skills $b_{y,i,t}$ receives a larger transfers from his parent, $\frac{k_{y,i,t}}{b_{y,i,t}} = \frac{1}{c}$. This in turns induces a change in his own transfer to his child, along the lines of the effects we just computed. Working in like manner we have that an increase in the parent's own second period cognitive skills $b_{o,i,t+1}$ leads from (38) to $\frac{\mathbf{k}_{y,t+1}}{b_{o,i,t+1}} = \frac{a}{c^2} G(\mathbf{s}_{o,t+1})_{coli}$, which leads in turn to a change in $s_{o,1,t+1}$, exactly as we analyzed earlier.

Social effects on the elasticity of intergenerational wealth transfers are generally not acknowledged by the literature. They are present when social networking is endogenous, but also when it is exogenous. The properties of the intergenerational wealth elasticity are summarized by Proposition 8, whose proof is in the Appendix.

Proposition 8. The elasticity of the transfer to the child, $k_{y,i,t+1}$, with respect to the transfer the parent herself received from her own parent, $k_{y,i,t}$, is given by

Part A.

$$EL(k)_t^{t+1} = \frac{a^2}{c^2} \left[\mathbf{G}(\mathbf{s}_{y,t})\mathbf{G}(\mathbf{s}_{o,t+1})\right]_{ii} \times$$

$$\frac{k_{y,i,t}}{\frac{a^2}{c^2} \left[\mathbf{G}(\mathbf{s}_{y,t})\mathbf{G}(\mathbf{s}_{o,t+1})\right]_{row\, i} \mathbf{k}_{y,t} + \frac{1}{c} b_{y,i,t+1} + \frac{a}{c^2} \left[\mathbf{G}(\mathbf{s}_{o,t+1})\right]_{row\, i} \mathbf{b}_{o,t+1} - \frac{1}{c\rho} \left[1 + \frac{a}{c} \mathbf{G}(\mathbf{s}_{o,t+1})_{row\, i} \mathbf{1}\right]}. \tag{46}$$

Part B.

$$0 < EL(k)_t^{t+1} < 1; \frac{\partial}{\partial k_{y,i,t}} EL(k)_t^{t+1} > 0. \tag{47}$$

Proposition 8 and Eq. (46) allow us to examine the model's prediction for the relationship between intergenerational wealth transfer elasticity and inequality. Corak (2013) popularized the so-called "Great Gatsby Curve" for a cross section of countries. The curve shows that across countries the intergenerational earnings elasticity increases with inequality. In particular, Corak (2013), Fig. 1, plots the intergenerational elasticity of earnings, against the Gini coefficient after taxes and transfers, for a number of OECD countries. It shows that the greater the inequality of earnings the greater the intergenerational elasticity and therefore the less the mobility in terms of earnings. The fit is not particularly tight, however popular the curve is, and thus allows for a host of other effects, in principle. Fig. 2 and 3, *ibid.,* show that in the United States, sons raised by top and bottom decile fathers are more likely to occupy the same position as their fathers. For sons of top (bottom) earning decile fathers, the probability that their sons' income fall in different deciles increases (decreases) with the income decile.

Intuitively, the larger is $\mathrm{EL}(k)_t^{t+1}$, the greater the inheritability of wealth transfers. Proposition 8, Part B, gives the exact dependence of $\mathrm{EL}(k)_t^{t+1}$, the elasticity to changes in the inequality in the components of $\mathbf{k}_{y,t}$. It is straightforward to show that for values of $k_{y,i,t}$ less (greater) than the mean, $\mathrm{EL}(k)_t^{t+1}$ decreases (increases) in the dispersion of $k_{y,i,t}$ around its mean, while holding the mean constant. Thus, at least when the coefficient of variation is used as a measure of inequality, the intergenerational wealth transfer elasticity decreases (increases) with inequality for wealth transfers less (greater) than the mean. Our prediction above that the elasticity is increasing in the transfer the parent herself receives is in agreement with the Great Gatsby Curve. The elasticity is decreasing in the child's cognitive skill when young and in the cognitive skill when old of the members of the parent's generation.

Englund *et al.* (2013) report empirical results in agreement with Proposition 8, Part B: estimated intergenerational wealth elasticities range between 0.296 and 0.410, across regressions of log five-year average child's wealth against the log of five-year average parents' wealth for different age groups [*ibid.,* Table A.3], and 0.497 and 0.530, across linear regressions [*ibid.,* Table 3].

Proposition 7, Part D, allows us to clarify the social effects on the marginal effect of increased initial wealth in a given period on that of descendants on initial wealth after more than one generation. Specifically, by applying (38) iteratively backwards, for τ periods, and under the assumption that the social network is fixed, we have that the coefficient of $\mathbf{k}_{y,t-\tau}$ in the expression for $\mathbf{k}_{y,t}$ is given by: $\mathbf{k}_{y,t} = \left[(a/c)^2\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)\right]^{t-\tau}\mathbf{k}_{y,t-\tau}$. In view of Proposition

7, Part D, this becomes:

$$k_{y,i,t} = \left[\left(\frac{a}{c} \right)^2 \frac{\overline{x^2}(\mathbf{s}_y)}{\overline{x}(\mathbf{s}_y)} \frac{\overline{x^2}(\mathbf{s}_o)}{\overline{x}(\mathbf{s}_o)} \right]^\tau \sum_\ell s_{y,\ell} s_{o,\ell} \sum_j \frac{s_{y,i}}{I\overline{x}(\mathbf{s}_y)} \frac{s_{o,j}}{I\overline{x}(\mathbf{s}_o)} k_{y,j,t-\tau}.$$

Thus, in view of (42), the effect of $k_{y,i,t-\tau}$ on $k_{y,i,t}$ attenuates geometrically; indeed, so do the effects of the human capitals of agents other than i, $k_{y,j,t-\tau}$, $j \neq i$. The attenuation is weaker, the greater the dispersion of socialization efforts, given that (42) holds. The greater the relative socialization effort, the coefficients of the $k_{y,j,t-\tau}$ in the summation above, the greater the effect of the respective human capital. Furthermore, socialization efforts themselves have a direct amplification effect via the term $\sum_\ell s_{y,\ell} s_{o,\ell}$. Similar arguments apply for the effects of cognitive skills, when young and when old.[17]

5.1.3 Moving across Neighborhoods over the Life Cycle

Suppose that individuals' life cycle consists of additional periods and that in principle individuals may move across sites. We may associate each period in an individual's life cycle with a different site, each of which is characterized by a different social interactions matrix $\mathbf{G}_\ell, \ell = 1, \ldots, L$. Each agent's i contribution to the social interactions matrix of each site consists of a row and of a column. Row $i, g_{ij}, j \neq i$, expresses the interactions effects from other agents; column $i, g_{ji}, j \neq i$, expresses the interactions effects on all other agents. The multiplicative structure of (38), where agents moving across sites is reflected on the coefficient of $\mathbf{k}_{y,t}$, which would now be made up of the product of the respective social interactions matrices, reflecting the effect of the three overlapping generations. Whereas the description of an individual's moving is somewhat unwieldy, the model is still helpful in tracking the evolution of the vector of human capitals for the entire economy as the social interactions matrix evolves exogenously. It would be interesting to generalize the model to account for deliberate choice of community, a topic that deserves attention in future research.

5.1.4 Steady States

The system of linear difference equations of section 5.1.1 could be examined further, and the first-order conditions for the social networking efforts studied in greater depth, especially if we were prepared to specify an exogenous process for the first- and second-period cognitive skills and while being cognizant of the stability analysis. We may also obtain more precise results by using both sets of first-order conditions at the steady state. However, a steady state analysis typically serves as an important benchmark, and we turn to that next.

Let us assume that the $b_{y,i,t}, b_{o,i,t}$ are time-invariant, and let us define

$$b^*_{y,i} \equiv b_{y,i} - \frac{1}{\rho}, \;\; b^*_{o,i} \equiv b_{o,i} - \frac{1}{\rho}.$$

Proposition 9 summarizes the results. The proof is in Appendix, Proofs.

<u>Proposition 9.</u> *The steady state solutions for human capitals* $(k_{y,i}, k_{o,i})$ *are defined in terms of the auxiliary variables* $\psi_y = \sum_{j\neq i} \frac{s_{y,j}k_{y,j}}{\sum_i s_{y,i}}, \psi_o = \sum_{j\neq i} \frac{s_{o,j}k_{o,j}}{\sum_i s_{o,i}}$.

Part A. The steady state solutions for human capitals $(k_{y,i}, k_{o,i})$ *satisfy*

$$k_{y,i} = \frac{b^*_{y,i}}{c - \rho a^2 \psi_o^2}, k_{o,i} = \frac{b^*_{o,i}}{c - \rho a^2 \psi_y^2}; \tag{48}$$

$$s_{y,i} = \rho a \psi_y \frac{b^*_{o,i}}{c - \rho a^2 \psi_y^2}, s_{o,i} = \rho a \psi_o \frac{b^*_{y,i}}{c - \rho a^2 \psi_o^2}, \tag{49}$$

where auxiliary variables (ψ_y, ψ_o) *satisfy:*

$$\psi_y = \frac{1}{c - \rho a^2 \psi_o^2} \frac{\mathbf{b}^*_y \cdot \mathbf{b}^*_o}{I\overline{x}(\mathbf{b}^*_o)}; \tag{50}$$

$$\psi_o = \frac{1}{c - \rho a^2 \psi_y^2} \frac{\mathbf{b}^*_y \cdot \mathbf{b}^*_o}{I\overline{x}(\mathbf{b}^*_y)}, \tag{51}$$

where $\mathbf{b}^*_y \cdot \mathbf{b}^*_o = \sum b^*_{y,i} b^*_{o,i}$.

Part B. If the vectors of cognitive skills $(\mathbf{b}_y, \mathbf{b}_o)$ *are not too asymmetric, the system of algebraic equations (50–51) may admit up to two sets of positive solutions, that define high-level and a low-level equilibria, from which the steady state values of human capitals and social networking efforts readily follow from (48–49).*

Thus, human capitals and networking efforts by young and old, $(k_{y,i}, k_{o,i}; s_{y,i}, s_{o,i})$, are uniquely defined in terms of the auxiliary variables (ψ_y, ψ_o) and parameters. They are associated with high-level and low-level equilibria. Human capitals $(k_{y,i}, k_{o,i})$ are proportional to their respective cognitive skills, $k_{y,i}$ to $b_{y,i}$, and $k_{o,i}$ to $b_{o,i}$), though with different factors of proportionality. In contrast, networking efforts, $(s_{y,i}, s_{o,i})$, are proportional to the cognitive skills corresponding to the life cycle period when individuals avail of them. That is, when individuals are old, and when their children are young, $(b_{o,i}, b_{y,i})$, again with different factors of proportionality. This simply reflect the timing conventions that have been incorporated in the model. Naturally, these solutions allow us again to express the optimum value of lifetime utility at a steady state for each dynasty as quadratic functions of $(b_{y,i}, b_{o,i})$, with the economy-wide distributions of the $(b_{y,i}, b_{o,i})$'s represented through the equilibrium values of (ψ_y, ψ_o). Note that in addition to the auxiliary functions $\overline{x}(\mathbf{b}_y^*), \overline{x}(\mathbf{b}_o^*)$ the cross-product $\mathbf{b}_y^* \cdot \mathbf{b}_o^*$ of first- and second-period cognitive skills also enter, indicating dependence on more complex moments of the distributions of cognitive skills.

Equations (50–51) have at most two solutions in (ψ_y, ψ_o), which can be characterized easily but not solved for explicitly. The steady state values of all endogenous variables then follow. Note that the life cycle model is crucial for the result: ψ_y and ψ_o would be equal to one another, were it not for the fact that, $b_{y,i} \neq b_{o,i}$, first-period and second-period cognitive skills are in general not equal to one another. Similarly, interesting complexity and accordant richness follow if cognitive skills may be influenced by means of investment, which I explore in section 5.4 further below.

If we were to assume, as in section 3.2, that the social networking efforts are given exogenously, in that case those of young and of old agents, with values not necessarily coinciding with the steady state ones, then a number of additional results are possible. First, under the assumption that the social networking efforts are constant over time, $(\mathbf{s}_y, \mathbf{s}_o)$, the system of equations (38–39) implies that a single equation for aggregate capital $\mathbf{k}_t = \mathbf{k}_{y,t} + \mathbf{k}_{o,t}$, may be obtained. The dynamics are exactly the same as in each of the two systems and no further discussion is necessary. Second, we may reformulate the evolution of human capitals in stochastic terms, as in the analysis of section 3.2.1 but now in terms of $(\mathbf{k}_{y,t}, \mathbf{k}_{o,t})$. Similar results regarding stochastic limits in the form of a power law are likely to be obtained.

Such results may be strengthened in the following way. Intuitively, as the number of overlapping generations increases, the matrix for human capitals in the laws of motion (38), (39), becomes the product of increasing number of factors. In the limit, as the number of overlapping generations tends to infinity, the product of stochastic matrices may be handled by techniques similar to those of section 3.2.1, leading to power laws.

5.2 Stochastic Shocks to Cognitive Skills

We turn next to the evolution of human capitals when the vector of cognitive skills, the $\mathbf{b}_{y,t}, \mathbf{b}_{o,t}$'s, is assumed to be stochastic.[18] This allows us to explore in greater depth the consequences of different specifications for the intergenerational dependence of skills in the presence of social connections. The economy evolves as follows: individual i after is born time t is endowed with cognitive skills, $b_{y,i,t}$, an exogenous state variable, and a wealth transfer from her parent $k_{y,i,t}$, an endogenous state variable whose evolution is described in detail below. Individual i avails herself of social interactions in exactly the same way as in the deterministic model above. I simplify the model by assuming that socialization efforts remain constant over time, $\mathbf{s}_o$ by the old, and $\mathbf{s}_y$ by the young, but will briefly explore the consequences of their endogeneity.

I assume that once $(b_{y,i,t}, k_{y,i,t})$ are realized at the beginning of time t, individual i's own second-period cognitive skills, denoted by $b_{o,i,t+1}$ and is to be realized in period $t+1$, is distributed conditional on $b_{y,i,t}$ according to $N(m_{o,i} + \frac{\sigma_o}{\sigma_y}\rho_o(b_{y,i,t} - b_{m,y,i}), \sigma_o^2(1-\rho_o^2))$. I assume that the cognitive skills of individual i's child, denoted by $b_{y,i,t+1}$ and is to be realized in period $t+1$, follows an $AR(1)$ process,

$$b_{y,i,t+1} = \bar{b}_{y,i} + \rho_b b_{y,i,t} + \epsilon_{y,i,t+1}, \tag{52}$$

where $\bar{b}_{y,i}$ is constant, and the stochastic shock $\epsilon_{y,i,t+1}$ is IID with distribution $N(0, \sigma_\epsilon^2)$. The unconditional distribution of $b_{y,i,t}$ is $N(b_{m,y,i}, \sigma_b^2)$, where $\sigma_b^2 = \frac{1}{1-\rho_b^2}\sigma_\epsilon^2$. Thus, conditional on $b_{y,i,t}$, $b_{y,i,t+1}$ is distributed according to $N((1-\rho_b)b_{m,y,i} + \rho_b b_{y,i,t}, \sigma_\epsilon^2)$, where $b_{m,y,i} = \frac{1}{1-\rho_b}\bar{b}_{y,i}$. Let $\mathbf{b}_m = (\mathbf{b}_{m,y}, \mathbf{b}_{m,o})$, with $(b_{m,y,i}, b_{m,o,i})$, as the components of the respective vectors. The unconditional variance-covariance matrix of $\mathbf{b}_{y,t}$ is $\sigma_b^2\mathbf{I}$. In view of the above assumptions, the conditional expectations $\mathcal{E}[b_{o,i,t+1}|b_{y,i,t}]$ and $\mathcal{E}[b_{y,i,t+1}|b_{y,i,t}]$ are known once $b_{y,i,t}$ is realized in the beginning of period t and as we see shortly, are sufficient to characterize the individual's decision problem.

Proposition 10. *Individual i chooses second period human capital and transfer to her child, $(k_{o,i,t+1}, k_{y,i,t+1})$, given the realization of $b_{y,i,t}$, and subject to uncertainty with respect to her own second period skills and her child's first period skills, $(b_{o,i,t+1}, b_{y,i,t+1})$.*

Part A. Defining the individual's decision problem of Proposition 7 under uncertainty yields the first-order conditions in vector form, the stochastic counterpart of (38–39):

$$\mathbf{k}_{y,t+1} = \frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)\mathbf{k}_{y,t} + \frac{1}{c}\mathcal{E}[\mathbf{b}_{y,t+1}|t] + \frac{a}{c^2}\mathbf{G}(\mathbf{s}_o)\mathcal{E}[\mathbf{b}_{o,t+1}|t] - \frac{1}{c\rho}\left[\mathbf{I} + \frac{a}{c}\mathbf{G}(\mathbf{s}_o)\right]\mathbf{1}, \quad (53)$$

$$\mathbf{k}_{o,t+1} = \frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_o)\mathbf{G}(\mathbf{s}_y)\mathbf{k}_{y,t} + \frac{1}{c}\mathcal{E}[\mathbf{b}_{o,t+1}|t] + \frac{a}{c^2}\mathbf{G}(\mathbf{s}_y)\mathcal{E}[\mathbf{b}_{y,t}|t] - \frac{1}{c\rho}\left[\mathbf{I} + \frac{a}{c}\mathbf{G}(\mathbf{s}_o)\right]\mathbf{1}, \quad (54)$$

Part B. Given social connections $(\mathbf{G}(\mathbf{s}_y), \mathbf{G}(\mathbf{s}_o))$, the state of the economy is described by the stochastic system for $(\mathbf{k}_{y,t}, \mathbf{b}_{y,t})$, where $\mathbf{k}_{y,t}$ evolves according to

$$\mathbf{k}_{y,t+1} = \frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)\mathbf{k}_{y,t} + \mathbf{G}_{adj,k}(\mathbf{s}_o)\mathbf{b}_{y,t} + \mathbf{C}_k, \quad (55)$$

$$\mathbf{k}_{o,t+1} = \frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)\mathbf{k}_{y,t} + \mathbf{G}_{adj,o}(\mathbf{s}_y)\mathbf{b}_{y,t} + \mathbf{C}_o, \quad (56)$$

where:

$$\mathbf{G}_{adj,y}(\mathbf{s}_o) = \frac{1}{c}\left[\rho_b\mathbf{I} + \rho_o\frac{\sigma_o}{\sigma_b}\frac{a}{c}\mathbf{G}(\mathbf{s}_o)\right];$$

$$\mathbf{C}_y = \frac{1}{c}\left[(1-\rho_b)\mathbf{I} - \frac{a}{c}\rho_o\frac{\sigma_o}{\sigma_b}\mathbf{G}(\mathbf{s}_o)\right]\mathbf{b}_{m,y} + \frac{a}{c^2}\mathbf{G}(\mathbf{s}_o)\mathbf{b}_{m,o} - \frac{1}{c\rho}\left[\mathbf{I} + \frac{a}{c}\mathbf{G}(\mathbf{s}_o)\right]\mathbf{1}; \quad (57)$$

$$\mathbf{G}_{adj,o}(\mathbf{s}_y) = \frac{1}{c}\left[\frac{\sigma_o}{\sigma_y}\rho_o\mathbf{I} + \frac{a}{c}\mathbf{G}(\mathbf{s}_y)\right];$$

$$\mathbf{C}_o = \frac{1}{c}\left[\mathbf{b}_{m,o} - \frac{a}{c}\rho_o\frac{\sigma_o}{\sigma_b}\mathbf{b}_{m,y}\right] - \frac{1}{c\rho}\left[\mathbf{I} + \frac{a}{c}\mathbf{G}(\mathbf{s}_o)\right]\mathbf{1}; \quad (58)$$

and $\mathbf{b}_{y,t}$ is an exogenous vector stochastic process, denoting first-period cognitive skills, introduced above.

Part C. Under the additional assumption that the vector of means and the variance-covariance matrix are time invariant, the stationary steady state is given by:

$$\mathbf{k}_y^* = \frac{1}{c}\left[\mathbf{I} + \frac{\left(\frac{a}{c}\right)^2 \cdot \frac{\overline{x^2}(\mathbf{s}_o)}{\overline{x}(\mathbf{s}_o)} \cdot \frac{\overline{x^2}(\mathbf{s}_y)}{\overline{x}(\mathbf{s}_y)}}{1 - \left(\frac{a}{c}\right)^2 \frac{\overline{x^2}(\mathbf{s}_o)}{\overline{x}(\mathbf{s}_o)} \cdot \frac{\overline{x^2}(\mathbf{s}_y)}{\overline{x}(\mathbf{s}_y)}}\mathbf{G}(\mathbf{s}_y;\mathbf{s}_o)\right]\left[\frac{1}{c}\mathbf{b}_{m,y} + \frac{a}{c^2}\mathbf{G}(\mathbf{s}_o)\mathbf{b}_{m,o} - \frac{1}{c\rho}\left[\mathbf{I} + \frac{a}{c}\mathbf{G}(\mathbf{s}_o)\right]\mathbf{1}\right]. \quad (59)$$

The deviation $\Delta\mathbf{k}_{y,t} = \mathbf{k}_{y,t} - \mathbf{k}_y^*$ *has a multivariate normal stationary limit distribution with mean* $\mathbf{0}$ *and variance covariance matrix* $\mathbf{\Sigma}_{y,\infty}$ *that satisfies:*

$$\mathbf{\Sigma}_{y,\infty} = \frac{a^4}{c^4}\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)\mathbf{\Sigma}_\infty\mathbf{G}(\mathbf{s}_o)\mathbf{G}(\mathbf{s}_y) + \mathbf{G}_{adj,y}(\mathbf{s}_o)\sigma_b^2\mathbf{I}\mathbf{G}^{\boldsymbol{T}}_{adj,y}(\mathbf{s}_o). \tag{60}$$

A necessary and sufficient condition for the existence of a positive definite matrix $\mathbf{\Sigma}_\infty$ *is that the matrix* $\frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)$ *be stable, for which by Proposition 7, Part C, a sufficient condition is that* $\frac{a^2}{c^2}$ *times the product of the largest eigenvalue of* $\mathbf{G}(\mathbf{s}_y)$ *and of* $\mathbf{G}(\mathbf{s}_o)$ *be less than 1. For the special case of (2) this condition is (42).*

If $\frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)$ *is stable, then:*

$$\mathbf{\Sigma}_{y,\infty} = \left[\mathbf{I} - \left(\frac{a}{c}\right)^4 \mathbf{G}(\mathbf{s}_y)^2\mathbf{G}(\mathbf{s}_o)^2\right]^{-1} \mathbf{G}_{adj,y}(\mathbf{s}_o)\mathbf{G}^{\boldsymbol{T}}_{adj,y}(\mathbf{s}_o)\sigma_b^2$$

$$= \left[\mathbf{I} + \frac{\left(\frac{a}{c}\right)^4 \cdot \frac{\overline{x^2}(\mathbf{s}_o)}{\overline{x}(\mathbf{s}_o)} \cdot \frac{\overline{x^2}(\mathbf{s}_y)}{\overline{x}(\mathbf{s}_y)}}{1 - \left(\frac{a}{c}\right)^4 \left[\frac{\overline{x^2}(\mathbf{s}_o)}{\overline{x}(\mathbf{s}_o)} \cdot \frac{\overline{x^2}(\mathbf{s}_y)}{\overline{x}(\mathbf{s}_y)}\right]^2}\mathbf{G}(\mathbf{s}_y;\mathbf{s}_o)\right] \mathbf{G}_{adj,k}(\mathbf{s}_o)\mathbf{G}^{\boldsymbol{T}}_{adj,k}(\mathbf{s}_o)\sigma_b^2, \tag{61}$$

where the matrix $\mathbf{G}(\mathbf{s}_y;\mathbf{s}_o)$ *is defined via its* (i,j) *element in (44).*

Part D. Under the above assumptions the vector the stationary steady state for $\mathbf{k}_{o,t}$ *satisfies*

$$\mathbf{k}_o^* = \left[\mathbf{I} + \frac{\left(\frac{a}{c}\right)^2 \cdot \frac{\overline{x^2}(\mathbf{s}_o)}{\overline{x}(\mathbf{s}_o)} \cdot \frac{\overline{x^2}(\mathbf{s}_y)}{\overline{x}(\mathbf{s}_y)}}{1 - \left(\frac{a}{c}\right)^2 \frac{\overline{x^2}(\mathbf{s}_o)}{\overline{x}(\mathbf{s}_o)} \cdot \frac{\overline{x^2}(\mathbf{s}_y)}{\overline{x}(\mathbf{s}_y)}}\mathbf{G}(\mathbf{s}_y;\mathbf{s}_o)\right]\left[\mathbf{k}_y^* + \frac{1}{c}\left[\mathbf{b}_{m,o} + \frac{a}{c}\mathbf{G}(\mathbf{s}_y)\mathbf{b}_{m,y} - \frac{1}{\rho}\left[\mathbf{I} + \frac{a}{c}\mathbf{G}(\mathbf{s}_o)\right]\mathbf{1}\right]\right]. \tag{62}$$

The deviation $\Delta\mathbf{k}_{o,t} = \mathbf{k}_{o,t} - \mathbf{k}_o^*$ *has a multivariate normal stationary limit distribution with mean* $\mathbf{0}$ *and variance covariance matrix* $\mathbf{\Sigma}_{o,\infty}$ *that is given by an expression as in (61), with* $\mathbf{G}_{adj,o}$*, defined in (58), in the place of* $\mathbf{G}_{adj,k}$*. where* $\sigma_o^2\mathbf{I}$ *denotes the variance covariance matrix of* $\mathbf{b}_o$*. A necessary and sufficient condition for the existence of a positive definite matrix* Σ_∞ *is that the matrix* $\frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)$ *be stable, for which by Proposition 7, Part C, a sufficient condition is that* $\frac{a^2}{c^2}$ *times the product of the largest eigenvalue of* $\mathbf{G}(\mathbf{s}_y)$ *and of* $\mathbf{G}(\mathbf{s}_o)$ *be less than 1. For the special case of (2) this condition is (42). If* $\frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)$ *is stable,* $\mathbf{\Sigma}_{o,\infty}$ *is given by the counterpart of (61) with* σ_o^2 *in the place of* σ_y^2*.*

Part E. The cross-sectional distribution of first-period human capitals at the stochastic steady-state, the $k_{y,i}$*'s, when we do not distinguish individuals' identities, is given by a* *mixture of univariate normals, with weights equal to I^{-1}, the relative proportion of agent types in the population (assuming for simplicity that $|\tau(i)| = I^{-1}$), with mean and variance given by:*

$$Mean_{k_y} = \frac{1}{I}\sum_i k^*_{y,i}, \ \ Var_{k_y} = \frac{1}{I}\sum_i (k^*_{y,i})^2 + \frac{1}{I} trace\left(\mathbf{\Sigma}_{y,\infty}\right) - \left(\frac{1}{I}\sum_i k^*_{y,i}\right)^2. \tag{63}$$

The respective cross-sectional distribution of second-period human capitals as well as that of the joint distribution of the first-period human capital individuals receive and the transfer they make to their children is obtained in like manner.[19]

Let us first discuss these results. The conditional expectations on the rhs of Eq. (53)-(54) are expressed in terms of $\mathbf{b}_{y,t}$, and are thus known once the $b_{y,i,t}$'s are realized. This allows us to solve out for the expectations and rewrite (53) in the form of (55). Furthermore, by using the envelope theorem in the derivations of Part A, the cognitive skill of an individual's child, $b_{y,i,t+1}$, which is realized at the beginning of period $t+1$, enters via its expectation only, while the inheritability parameter do enter the derivations. Thus the resulting Eq. (55), which is stochastic, may be solved in the standard fashion for such stochastic equations, which is accomplished by Part C above.

The properties of $\mathbf{G}(\mathbf{s}_y), \mathbf{G}(\mathbf{s}_o)$ are crucial determinants of the properties of the means, $(\mathbf{k}^*_y, \mathbf{k}^*_o)$ and of the variance-covariance matrices of the limit distribution, $\mathbf{\Sigma}_{y,\infty}, \mathbf{\Sigma}_{o,\infty}$. It is straightforward to generalize the above results if different individuals' cognitive skills, the components of $(\mathbf{b}_{y,t}, \mathbf{b}_{o,t})$ are not independent and identically distributed draws from the same distribution. It is interesting that even if the components of $(\mathbf{b}_y, \mathbf{b}_o)$ are not independent and identically distributed random variables, the variance covariance matrices of human capitals display a lot of richness, on account of the social interactions structure, when it is exogenous. In view of (44), the vectors of cognitive skills are multiplied by $s_{y,i}s_{o,j}$ in the expressions for steady state human capitals. When the social interactions structure is endogenous, the fact that human capitals are proportional to their respective cognitive skills vectors, $(\mathbf{b}^*_y, \mathbf{b}^*_o)$, constitutes an important benchmark for the analysis.

Two particularly notable features of Part E, Proposition 10, are: one, even though the $k_{y,i}$'s are correlated, the variance of the cross-sectional distribution does not depend on the pattern of correlations beyond what is reflected on the trace $(\boldsymbol{\Sigma}_{y,\infty})$. The latter does reflect full dependence on the social structure. And two, the cross-sectional distribution as a convex combination of normal densities might exhibit thick tails and might not be unimodal. I note that in (63), in addition to the determinants of $\mathbf{k}_y$, the other new element is the trace of $\mathbf{G}(\mathbf{s}_y;\mathbf{s}_o)$, which from (44) involves a new term:

$$\frac{\left(\sum_\ell s_{y,\ell}s_{o,\ell}\right)^2}{I\overline{x}(\mathbf{s}_y)I\overline{x}(\mathbf{s}_o)}\,.$$

Therefore, both then mean and variance increase with heterogeneity in social connection efforts, as measured by $\frac{\overline{x^2}(\mathbf{s}_o)}{\overline{x}(\mathbf{s}_o)}\cdot\frac{\overline{x^2}(\mathbf{s}_y)}{\overline{x}(\mathbf{s}_y)}$, other things being equal. But the expression for the variance includes an additional effect, the correlation of first- and second-period social interactions efforts in the numerator of the above expression. Thus, the greater this correlation, *cet. par.*, the greater the variance of the cross-sectional distribution at the steady state.

In sum, the different forces determining the dispersion of human capital at the steady state are summarized neatly in (60) and are of course reflected in the expression for the variance of the cross-sectional distribution (63). The first factor in the rhs of (60) is only a function of the properties of the social interactions part of the model. The second factor[20] $\mathbf{G}_{\text{adj,k}}(\mathbf{s}_o)\mathbf{G}^{\mathbf{T}}_{\text{adj,k}}(\mathbf{s}_o)$ reflects the interaction of the correlation coefficient ρ_o between first- and second-period own cognitive skills with the second-period social interactions matrix, and that between first-period cognitive skills and those of the child, ρ_b, which enters directly and independently of social interactions. The third is the variance of the shock σ_b^2 in the $AR(1)$ structure expressing intertemporal evolution of individual cognitive shocks. To the best of my knowledge this decomposition is a new finding.

5.3 Stochastic Shocks to Non-Cognitive Skills

James Heckman and his collaborators have argued that economists can help decisively in establishing quantitatively the role of non-cognitive skills on par with cognitive skills in human development. See Heckman (2008). Economists often concentrate on the so-called "Big Five", abbreviated as *OCEAN*.[21] It is understood that these factors represent personality traits at the broadest level of abstraction, and summarize a large number of distinct, more specific personality traits, all of which are subject to intensive research by psychologists and now by economists. as well.[22] In an admittedly cavalier manner, I adopt (for the purpose of exposition) the convention that the social interactions coefficient a which expresses the value an individual attaches to social interactions measures non-cognitive skills, in the sense of an individual's ability to benefit from social interactions reflects personality traits.

I redefine the individual's decision problem to individualize a as $(a_{y,i,t}, a_{o,i,t+1})$ and assume them to be random variables. We redefine the value functions of Proposition 7, Part A, when choosing $(k_{o,i,t+1}, k_{y,i,t+1})$, and consequently the individual treats as uncertain the value that she would derive from $k_{o,i,t+1}$ in her second period of her own life and the value accruing to her child from the transfer $k_{y,i,t+1}$. The former depends on the human capitals of others, $k_{y,j,t}, j \neq i$, which are known when she makes the decision at time t, but the effect depends on the realization of $a_{y,i,t+1}$. The latter depends on the cognitive skills of the child at time $t+1$ and the realization of the social interactions effect $a_{o,i,t+2}$ at time $t+2$. The results are summarized by Proposition 11; the proof is immediate.

Proposition 11. *Under the assumption that the social interactions coefficients in the problem defined by Proposition 7, Part A, is a random variable, the definitions of the value functions, as in Propositions 7 and 10, for individual i as of time t and for her child as of time $t+1$, $\mathcal{V}_i^{[t]}(k_{y,i,t}, \mathbf{s}_{o,t}; a_{i,t})$, $\mathcal{V}_i^{[t+1]}(k_{y,i,t+1}, \mathbf{s}_{o,t+1}; a_{i,t+1})$, that are associated with an individual's lifetime utility when he is young at t and when he is old at $t+1$, are modified accordingly and given in the Appendix.*

Under the assumption that that the networking efforts are constant, $\mathbf{s}_{y,t} = \mathbf{s}_y, \mathbf{s}_{o,t} = \mathbf{s}_o$, the first-order conditions for human capitals $(k_{y,i,t+1}, k_{o,i,t+1})$ in vector form are:

$$\mathbf{k}_{y,t+1} = \frac{\bar{\mathbf{A}}_{y,t+1}\bar{\mathbf{A}}_{o,t+1}}{c^2}\mathbf{G}(\mathbf{s}_{y,t})\mathbf{G}(\mathbf{s}_{o,t+1})\mathbf{k}_{y,t} + \frac{1}{c}\mathbf{b}_{y,t+1} + \frac{\bar{\mathbf{A}}_{y,t+1}}{c^2}\mathbf{G}(\mathbf{s}_{o,t+1})\mathbf{b}_{o,t+1} - \frac{1}{c\rho}\left[\mathbf{I} + \frac{\bar{\mathbf{A}}_{y,t+1}}{c}\mathbf{G}(\mathbf{s}_{o,t+1})\right]\mathbf{1}. \tag{64}$$

$$\mathbf{k}_{o,t+1} = \frac{\bar{\mathbf{A}}_{y,t+1}\bar{\mathbf{A}}_{o,t+1}}{c^2}\mathbf{G}(\mathbf{s}_{o,t})\mathbf{G}(\mathbf{s}_{y,t})\mathbf{k}_{o,t} + \frac{1}{c}\mathbf{b}_{o,t+1} + \frac{\bar{\mathbf{A}}_{y,t}}{c^2}\mathbf{G}(\mathbf{s}_{y,t})\mathbf{b}_{y,t} - \frac{1}{c\rho}\left[\mathbf{I} + \frac{a}{c}\mathbf{G}(\mathbf{s}_{y,t})\right]\mathbf{1}, \tag{65}$$

where $\bar{\mathbf{A}}_{y,t+1}$, $\bar{\mathbf{A}}_{o,t+1}$ denote the diagonal matrices composed of the conditional means

$$\mathcal{E}[a_{y,i,t+1}|t],\ \mathcal{E}[a_{o,i,t+1}|t].$$

Part B. If $\bar{\mathbf{A}}_{y,}, \bar{\mathbf{A}}_{o,t}$ are time invariant, sufficient conditions for the existence of meaningful steady state values of $(\mathbf{k}_y, \mathbf{k}_o)$ amount to sufficient conditions for the invertibility of

$$\mathbf{I} - c^{-2}\bar{\mathbf{A}}_y\bar{\mathbf{A}}_o\mathbf{G}(\mathbf{s}_o)\mathbf{G}(\mathbf{s}_y), \tag{66}$$

namely that the product of the largest $\frac{a_{y,i}a_{o,i}}{c^2}$ times the largest eigenvalues of each of the matrices $\mathbf{G},(\mathbf{s}_o)\mathbf{G}(\mathbf{s}_y)$ be less than 1.

Allowing for a stochastic non-cognitive shock via parameters $a_{i,t}$'s does not change substantially the first-order conditions. The linear-quadratic nature of the problem makes for only the conditional means to enter, and the difference from the deterministic case is noteworthy only if the random variables $a_{i,t}$ were not IID over individuals and time. E.g., if $a_{i,t}$ is serially correlated over time, the system of equations (64–65) becomes stochastic. It is also conceptually straightforward to allow for correlation between cognitive and non-cognitive shocks, that is between $a_{i,t}$ and first-period cognitive skills, $b_{y,i,t}$, and therefore with $(b_{y,i,t+1}, b_{o,i,t+1})$, as well. Such a generalization may be accommodated by the tools employed by Proposition 10. Although the derivations would not be trivial extensions of Proposition 10, they are tractable. The steady state means and variance covariance matrix would reflect the stochastic dependence parameters between the stochastic processes for $\mathbf{b}_{y,t}, \mathbf{a}_{y,t}$. The fact that incorporating stochastic variation in non-cognitive skills (or, social competence) is fairly tractable is good news from the viewpoint of seeing the impact of all three possible sources of variation of human capitals across the population. Although solving (64)–65) for the steady states is no longer so straightforward as before, the three sources of variation are clear. For both the $k_{y,i}$'s and the $k_{o,i}$'s, the respective period autarkic solution, $\frac{1}{c}b_{y,i}$ is augmented by means of a component that reflects social interactions in both periods multiplicatively, adjusted by the mean non-cognitive skills, and a component that reflects $\frac{1}{c}b_{o,i}$, adjusted by the social interactions weights associated with the second period in individuals' lifetimes and by the mean non-cognitive skills. Thus, individuals' non-cognitive skills have spillovers on other individuals' behavior. The expressions for the steady-state solutions are little simplified if we assume that the mean non-cognitive effects and the social interaction weights are time invariant and equal across first- and second periods of individuals's lifetimes.

Furthermore, the stochastic structure for the $(\mathbf{a}_{y,t}, \mathbf{a}_{o,t+1})$ may be generalized to allow for persistent heterogeneity and random variation in each period. This would allow one to compare the empirical performance of such extensions of the model with alternative formulations that allow for amplification of social interactions effects either intergenerationally, as suggested by the results of Lindahl et al. (2015), or within and across social groups, as elaborated by Ioannides and Loury (2004) and Calvó-Armengol and Jackson (2004).

5.4 Investment in Cognitive Skills in a Model of Two Overlapping Generations with Two Subperiods Each

We reformulate the model to allow individuals to use resources to influence the cognitive skills of their children, while we retain the feature that their social networking decisions also influence their children's social networks, via the social structure which influences the child but results from parents' decisions. We continue to interpret the latter as influence via non-cognitive skills. We retain the overlapping generations structure and assume that youth and adulthood lasts for two subperiods each, early youth and youth, and adulthood and old age, respectively. Here t indexes subperiods. So, an adult at time t, who was born at time $t-2$ and is in her third subperiod of her life, gives birth to a child. The child lives for four subperiods, $t, t+1, t+2, t+3$, during two of which she overlaps with the parent who is still alive, and then lives on for two more subperiods. She in turn gives birth to her own children at time $t+2$, when she herself is an adult. Individuals make decisions affecting the household only in adulthood and old age. For a child born at time t, her cognitive skills when she become an adult at time $t+2$ are determined[23] by the given input at birth, $b_{y,i,t}$, which may be constant, and investments $(\iota_{c1,t}, \iota_{c2,t+1})$:

$$b_{y,i,t+2} = b_{o,i,t+3} = \beta_0 b_{y,i,t} + \beta_1 \iota_{c1,t} + \beta_2 \iota_{c2,t+1}, \tag{67}$$

where $\beta_0, \beta_1, \beta_2$ are positive parameters, and $(\iota_{c1,t}, \iota_{c2,t+1})$ are resource costs, which are incurred, contemporaneously with the respective adjustment costs, in time periods t, and $t+1$, the first and second subperiods in a child's life time, $\frac{1}{2}\gamma_1 \iota_{c1,t}^2$, $\frac{1}{2}\gamma_1 \iota_{c2,t+1}^2$, respectively. Investments $(\iota_{c1,t}, \iota_{c2,t+1})$ as decision variables are part of the individual's life cycle optimization.

Proposition 12. For an individual born at t, cognitive skills and human capital in period

t are given, $(b_{y,i,t}, k_{y,i,t})$; she benefits from the networking efforts of the parents' generation, $\mathbf{s}_{o,t-1}$, who are in the third subperiod of their lives. She chooses at time t her own second subperiod human capital and the first subperiod transfer to her own child at time $t+2$, respectively $\{k_{o,i,t+1}, k_{y,i,t+2}\}$; and the first and second subperiod networking efforts, $\{s_{y,i,t,}, s_{o,i,t+1}\}$, respectively. She benefits herself in her own second subperiod and her child benefits when the child is in her first subperiod of her life and the parent herself in her third subperiod of her life. The adjustment costs for decisions $\{s_{y,i,t,}, k_{o,i,t+1}\}$, are incurred in period t. The optimization problem treats the cognitive skills, $b_{y,i,t+2}$, of the individual's child and the transfer she receives when she becomes an adult, $k_{y,i,t+2}$, as being determined simultaneously.

Part A. The first order conditions for $(\iota_{c1,t}, \iota_{c2,t+1})$ yield:

$$b_{y,i,t+2} = b_{o,i,t+3} = \beta_0 b_{y,i,t} + \rho\rho_\beta[k_{y,i,t+2} + \rho k_{o,i,t+3}] - \rho_\beta, \tag{68}$$

where parameter ρ_β is defined as $\rho_\beta \equiv \left(\rho\frac{\beta_1}{\gamma_1} + \frac{\beta_2}{\gamma_2}\right)$.

Part B. The first-order conditions with respect to $(\mathbf{k}_{y,t+2}, \mathbf{k}_{o,t+2})$ yield a first-order linear difference system in $\mathbf{k}_{o,t+2}$:

$$\mathbf{k}_{o,t+3} = \mathbf{b}_{\mathit{eff}} + \frac{a}{\rho^* c}\mathbf{G}(\mathbf{s}_{y,t+2})\left[\mathbf{I} - \frac{\hat{a}}{c}\mathbf{G}(\mathbf{s}_{y,t+2})\right]^{-1}\frac{a}{\tilde{\rho}c_{cs}}\mathbf{G}(\mathbf{s}_{o,t+2})\mathbf{k}_{o,t+2}, \tag{69}$$

where $\rho^* \equiv 1 - \frac{\rho^2\rho_\beta}{c}$, $\tilde{\rho} \equiv 1 - \frac{\rho\rho_\beta}{c_{cs}}$, $c_{cs} \equiv c - \rho\rho_\beta$, and $\hat{a} \equiv \frac{a\rho^2\rho_\beta}{\rho^*\tilde{\rho}c_{cs}}\left(1 - \frac{\rho^3\rho_\beta^2}{\rho^*\tilde{\rho}cc_{cs}}\right)^{-1}$, and $\mathbf{b}_{\mathit{eff}}$ are constant. The optimal $\mathbf{k}_{y,t+2}$ follows from $\mathbf{k}_{o,t+2}$ according to:

$$\mathbf{k}_{y,t+2} = \left[\mathbf{I} - \frac{\hat{a}}{c}\mathbf{G}(\mathbf{s}_{y,t+2})\right]^{-1}\left[\mathbf{b}'_{\mathit{eff}} + \frac{a}{\tilde{\rho}c_{cs}}\mathbf{G}(\mathbf{s}_{o,t+2})\mathbf{k}_{o,t+2}\right], \tag{70}$$

where $\mathbf{b}'_{\mathit{eff}}$ is a constant.

Part C. The stability of (69) rests on the spectral properties of

$$\frac{a}{\rho^* c}\,\frac{a}{\tilde{\rho}c_{cs}}\mathbf{G}(\mathbf{s}_{y,2})\mathbf{G}(\mathbf{s}_{o,2})\left[\mathbf{I} + \frac{\hat{a}}{c}\frac{1}{1 - \frac{\hat{a}}{c}\frac{\overline{x^2}(\mathbf{s}_{y,2})}{\overline{x}(\mathbf{s}_{y,2})}}\mathbf{G}(\mathbf{s}_{y,2})\right], \tag{71}$$

provided that $\frac{\hat{a}}{c}\frac{\overline{x^2}(\mathbf{s}_{y,2})}{\overline{x}(\mathbf{s}_{y,2})} < 1$. A sufficient condition for the stability of (69) is that $\frac{a}{\rho^* c}\,\frac{a}{\tilde{\rho}c_{cs}}$ times the product of the maximal eigenvalue of $\mathbf{G}(\mathbf{s}_{y,2})$ and of $\mathbf{G}(\mathbf{s}_{o,2})$ times 1 plus the maximal eigenvalue of $\frac{\hat{a}}{c}\frac{1}{1-\frac{\hat{a}}{c}\frac{\overline{x^2}(\mathbf{s}_{y,2})}{\overline{x}(\mathbf{s}_{y,2})}}\mathbf{G}(\mathbf{s}_{y,2})$ be less than 1.

It follows that the first-order condition for $k_{y,i,t+2}$ must reflect the influence that decision has, as implied by the optimization problem, on $b_{y,i,t+2}$. Since $b_{y,i,t+2} = b_{o,i,t+3}$ the utility per period from the last two subperiods of the child's lifetime contribute to the first-order conditions.

In a notable difference from the previous model, we now see a key new role for the social networking that individuals avail of when young. The product $\mathbf{G}(\mathbf{s}_{y,t+2})\mathbf{G}(\mathbf{s}_{o,t+2})$ is adjusted by $\left[\mathbf{I} - \frac{\hat{a}}{c}\mathbf{G}(\mathbf{s}_{y,t+2})\right]^{-1}$. Intuitively, this effect acts to reinforce the effects of social networking when the child is young and in her first subperiod of the child's life. This readily follows from (69) and (70) above and may be simplified by using the results of Proposition 7, Part D. Feedbacks are generated due to the investment in cognitive skills.

It is important to recognize that the derivation of (69), as well as those of (38) and (39) earlier, do not make use of first-order conditions for the social connections. Therefore, if we were to expand the number of overlapping generations, then the system of linear equations in the human capitals is updated iteratively and links the initial and final human capital vectors by means of the product of the social interactions matrices associated with each intervening generation. Thus, in an extension of the model where individuals may move across communities and avail themselves of different social interactions in different communities, the impact of residential histories is reflected on the product of the respective matrices. As indicated earlier in section 5.1.3 above, it would be interesting to address in future research the equilibrium outcome for an entire economy when individuals make deliberate decisions about community choice.

6 Conclusions

The dynamic models analyzed by this paper offer a novel view of the joint evolution of human capital investment and social networking. Those of the models that are embedded in overlapping generations frameworks inherit the full potential of that workhorse of modern growth theory and macroeconomics. The analysis first takes advantage of formal similarities between infinite horizon dynastic life cycle modeling and overlapping generations models with intergenerational transfers. The dynamic models of the paper share the important feature namely that individuals' lifetime human capital accumulation plans are distinguished from intergenerational transfers, while allowing for an endogenous social structure. The model where endogenous investment influences the cognitive skills of one's child is analytically considerably more complicated than when cognitive skills are given, however, because of additional dynamic complexity. In our basic model with overlapping generations, individuals receive a transfer from their parents in the first period of their lives and avail themselves of the social connections that their parents chose at that same period. They in turn choose their own second-period human capital, own second-period social connections, and transfer to their children. The dynamical system involving the vectors of life cycle accumulation and transfers, given the social network, is still linear in those magnitudes and tractable. The endogeneity of the social structure makes that analysis quite more complicated but considerably richer. Yet, the tools of the paper do allow us to study the underlying steady states for individuals' life cycle accumulation, intergenerational transfers, and social connections for themselves and for their children in great detail. The elasticity of the intergenerational transfer received by an individual is increasing in the intergenerational transfer received by the parent, exhibits rich dependence on social effects, and is positive and less than 1. The dynamics of demographically increasingly complex models are shown to be tractable. The effects of stochastic shocks to cognitive as distinct from non-cognitive skills are studied by means of a novel interpretation of individual preferences with social interactions. The stochastic steady state analysis allows us to study the cross-section steady-state human capital distribution in the presence of shocks to cognitive skills. The paper offers a novel view of the consequences for inequality of the joint evolution, endogenous or exogenous, of social connections and human capital investments. It allows for intergenerational transfers of both human capital and social networking endowments in dynamic and steady-state settings of dynastic overlapping-generations models of increasing demographic complexity. Intergenerational transfer elasticities exhibit rich dependence on social effects. The paper highlights the separable effects on human capital dispersion of social interactions alone, as distinct from the joint effects of the intertemporal evolution of skills. To the best of my knowledge this decomposition is a new finding. The dynamics of demographically increasingly complex models are shown to be tractable. Their stochastic steady states allows us to study the cross-section human capital distribution in the presence of shocks to underlying parameters that may be interpreted as shocks to cognitive and non-cognitive skills.

Interestingly, the consequences for inequality of the endogeneity of social connections are underscored by examining our models when social connections are assumed to be exogenous. When social connections are an outcome of ad hoc decision making and not optimized, individuals' human capital reflect an arbitrarily more general dependence on social connections across individuals. The dependence does not reduce to aggregate statistics and highlights both "whom you know" and "what you know" in the determination of individual human capitals and their steady-state distribution. When individuals optimize over their social connections, their actions make up for the arbitrariness of outcomes and thus reduce dependence to a smaller set of fundamentals.

There are many aspects of the present paper that deserve further attention in future research. To name a few, in addition to the need to deal with the equilibrium selection problem and to develop more general stochastic formulations, one would be to fully explore the interfaces between network formation and neighborhood choice, where one must also account for the costs associated with clustering to attractive neighborhoods; another would be to allow individuals to learn from others' social competence and to introduce a firmer link with the job market; yet another would be to examine how the network formation process might be influenced by public policy. Modeling explicitly the acquisition of cognitive and non-cognitive skills as a joint process and their importance as components of jobs also appears to be interesting. Although no general theory of network formation is available, endogenous networks may be defined for those different classes of problems, all of which bear upon the emergence of inequality.

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Notes

[1]**Disclosure.** Tufts University is the only source of support for this paper.

[2]Albornoz, Cabrales, and Hauk (2014) develop a conceptually similar use of the Cabrales *et al.* model, but in a static context.

[3] This basic model may be augmented to account for a variety of motivations, such as altruism, conformism and habit formation. See Ioannides (2013), Ch. 2.

[4]See Goldberger (1989) for a skeptical view of some of Becker and Tomes' specific predictions. Goldberger also welcomes broadly behavioral predictions obtained by sociologists but not necessarily emanating from utility maximization. He hints that sociological predictions that strong intergenerational links for socioeconomic status may be understated by economists' focus on intergenerational effects on income and its impact on inequality.

[5]Cabrales *et al.* follow standard practice in this literature and define a finite number of types of players and work with an $m-$replica game, for which the total number of individuals is a large multiple of the number of types. In this fashion, as we see further below, it is possible to increase the number of individuals in order to reduce the influence of any single one of them and be able to characterize outcomes in a large economy. *Ibid.*, p. 341.

[6]In $(\varpi, \vartheta)-$ space, the tangent from the origin to the graph of (14) must have slope less than $\tilde{a}^{-1}$.

[7]Formulations of determinants of interactions with rich demographics may be helpful in accommodating the range of empirical issues broached by Ioannides and Loury (2004).

[8]See the discussion in Cabrales *et al.*, p. 351. As they argue, this can help to explain why, "in different locales, children whose parents have similar characteristics (e.g. income, education level) or are similarly talented as other children (say, measured by I.Q.) end up having very different educational outcomes or different levels of parental educational efforts."

[9]The so-called CES structure is in turn a special case of a mean value with an arbitrary function [Hardy *et al.* (1952), p. 65]. That is, let $y(k)$ be a function, which is assumed to be continuous and strictly monotonic, in which case so is its inverse, $y^{-1}(k)$. The CES structure defined here is simply $y^{-1}\left(\sum gy(k)\right)$, for $y(k) = k^{1-\frac{1}{\xi}}$.

[10]In fact, a feature such as the last one is relied upon by Lucas and Moll (2014), where individuals divide their time between producing goods using their existing knowledge and interacting with others in search of new productivity-enhancing ideas. Such interactions take the form of pairwise meetings, which is simply an opportunity for each individual to observe the productivity of someone else. If that is higher than his own, he adopts it in place of the one he came in with. To ensure that the growth generated by the process is sustained, Lucas and Moll assume that the stock of good ideas to be discovered is inexhaustible. It is possible to introduce this set of possibilities once we have allowed for shocks that in effect renew the set of productive ideas.

[11]The case of cognitive shocks that are independent and conditionally identically Fréchet-distributed random variables is discussed in Appendix A.

[12] In a nutshell, one may work with the properties of minimum of a set of random variables $\min_i\{\psi_i\}_{i\in\mathcal{I}}$ via $-\max_i\{-\psi_i\}_{i\in\mathcal{I}}$. Such a treatment may rely on the properties of the reverse-Weibull class of distributions whose distribution function, given by $\exp[-(-\psi)^{\chi}]$, for $x < 0$, and equal to 1, for $x \geq 0$, has positive support only over the negative half axis of the real line. See De Haan and Ferreira (2006).

[13]See Ioannides and Soetevent (2007) who assume preferences that accommodate more general social effects than those allowed for here. For example, when a conformist global effect is present, modeled by individuals' suffering disutility from the gap between own human capital and the lagged average human capital in the economy, and coexists with local effects, expressed in terms of comparison of an individual's outcome with those of his social contacts, the model involves expectations of individuals' future actions. The resulting system of second-order difference equations with expectations may be characterized. See *ibid.,* Proposition 4. We leave for future research the consequences of such assumptions for endogenous social structures.

[14]This argument is reminiscent of arguments explaining the emergence of power laws elsewhere in the economics literature. See for the city size distribution case Ioannides (2013), Ch. 8.

[15]The convergence in distribution of $\tilde{\mathbf{G}}(\mathbf{\Phi}_1)\cdots\tilde{\mathbf{G}}(\mathbf{\Phi}_t),\ t\to\infty$, to a non-zero matrix is of independent interest and may be ensured under appropriate and not very restrictive conditions. See Kesten and Spitzer (1984).

[16]In fact, Samuelson (1958) itself is cast in terms of three-overlapping generations. Azariadis, Bullard and Ohanian (2004) find additional properties in economies with many overlapping generations, in particular with respect to monotonicity (or non-monotonicity) of the equilibrium price when consumptions in different periods are weak gross substitutes.

[17]Of course, the above expression is not the complete solution one obtains by iterating backwards. Terms associated with cognitive effects also matter.

[18]The case of non-cognitive skills is examined further below.

[19]I thank Vassilis Hajivassiliou for his help with the proof of Part E, Proposition 10.

[20]This expression becomes $\mathbf{G}_{\text{adj,k}}(\mathbf{s}_o)\mathbf{\Sigma}_b^2\mathbf{G}_{\text{adj,k}}^{\mathbf{T}}(\mathbf{s}_o)$, if the shocks in the AR(1) structure (52) are correlated and have a full variance-covariance matrix $\mathbf{\Sigma}_b^2$.

[21]These are: *O*penness (curiosity and receptivity), *C*onscientiousness (including being well organized and efficient), *E*xtraversion (including friendliness and whether one is high energy), *A*greeableness (including friendliness and compassion), and *N*euroticism (which includes self-confidence and sensitivity to stress).

[22]In a key contribution to the empirical literature on skill formation, Cunha, Heckman and Schennach (2010) estimate the technology of cognitive and non-cognitive skill formation. They allow for adult outcomes to develop from a multistage process, where cognitive and non-cognitive skills in each stage are produced (by means of CES production functions) by cognitive and non-cognitive skills at the preceding stage along with investment and the cognitive and non-cognitive skills of an individual's parents.

[23]The assumption of infinite substitutability of investments in cognitive skills according to (67) is a limit case of the assumption of finite substitutability by Heckman and Mosso (2014), and is made for analytical convenience.

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8 Appendix A: Proofs

8.1 Proposition 1. Proof

It readily follows from (10) that the necessary conditions imply that $\frac{s_i}{k_i}$ is independent of i. Let the common ratio be

$$\frac{s_i}{k_i} = \varpi. \tag{72}$$

With the notation introduced in (4) above, the auxiliary term $\lambda_{a/c}$ in (11) becomes, using (10):

$$\lambda_{a/c} = \frac{a}{c} \frac{\bar{x}(\mathbf{s})}{\bar{x}(\mathbf{s}) - \frac{a}{c}\overline{x^2}(\mathbf{s})} = \frac{a}{c - \varpi^2}.$$

In view of these results, (11) is simplified as follows:

$$ck_i = b_i + \frac{a}{c - \varpi^2} s_i \frac{\sum_{j=1}^{I} b_j s_j}{I\bar{\mathbf{s}}}. \tag{73}$$

Using the previous results with the equation, it follows that k_i/b_i is constant,

$$\frac{k_i}{b_i} = \vartheta. \tag{74}$$

Thus,

$$c\vartheta = 1 + \vartheta \frac{a}{c - \varpi^2} \frac{\overline{x^2}(\mathbf{b})}{\overline{x(\mathbf{b})}},$$

Recalling the definition of $\tilde{a}$ in (3) above, (10) becomes:

$$\varpi = \tilde{a}\vartheta. \tag{75}$$

This allows us to write the above condition as:

$$\vartheta = \frac{1}{c - \varpi^2}. \tag{76}$$

The system of equations (75–76) define the solution (ϖ^*, ϑ^*), to the multi-person game. The solutions for (k_i, s_i) follow:

$$k_i^* = \vartheta^* b_{\tau(i)}, \quad s_i = \varpi^* \vartheta^* b_{\tau(i)}. \tag{77}$$

Q.E.D.

8.2 Proposition 2. Proof

For this case, the first-order conditions are:

$$b_i + as_i \min_{j \neq i}\{k_j\} - ck_i = 0; \tag{78}$$

$$ak_i \min_{j \neq i}\{k_j\} - s_i = 0. \tag{79}$$

This leads a system of two equations, just as before:

$$\vartheta^{-1} + a\varpi \min_{j \neq i}\{k_j\} = c; \tag{80}$$

$$a \min_{j \neq i}\{k_j\} = \varpi. \tag{81}$$

Disregarding the imprecision that $\min_{j \neq i}\{k_j\} = \min_{j \in \mathcal{I}}\{k_j\}$ we have that $\min_{j \in \mathcal{I}}\{k_j\} = \vartheta \min_{j \in \mathcal{I}}\{b_j\} = b_{\min}$. Thus,

$$\varpi = ab_{\min}, \ \vartheta = \frac{1}{c - (ab_{\min})^2},$$

and solutions (17 follow. Q.E.D.

8.3 Proposition 3. Proof

8.3.1 The Best Individual is *the* Role Model"

For this case, the first-order conditions are:

$$b_i + as_i \max_{j \neq i}\{k_j\} - ck_i = 0; \tag{82}$$

$$ak_i \max_{j \neq i}\{k_j\} - s_i = 0; \tag{83}$$

By substituting for s_i from (83) into (82), the resulting equations are defined solely in terms of $\mathbf{k}$ as fixed points of:

$$k_i = \frac{b_i}{c_i} \frac{1}{c - a^2 \left(\max_{j \neq i}\{k_j\}\right)^2}, i \in \mathcal{I}.$$

By working in like manner as above, we have that:

$$\varpi = ab_{\max}, \ \vartheta = \frac{1}{c - (ab_{\max})^2},$$

and $k_j = b_j \frac{1}{c-(ab_{\max})^2}$, $s_j = b_j \frac{ab_{\min}}{c-(ab_{\max})^2}$. and solutions (18 follow. Q.E.D.

8.4 Proposition 4. Proof

For this case, the first-order conditions are:

$$\psi_i + as_i\ E_{\psi_{j|\psi_i}} \max_{j\neq i}\{k_j(\psi_j)\} - ck_i = 0; \tag{84}$$

$$ak_i\ E_{\psi_{j|\psi_i}} \max_{j\neq i}\{k_j(\psi_j)\} - s_i = 0; \tag{85}$$

By substituting for s_i from (85) into (84), we get:

$$\frac{k_i(\psi_i)}{\psi_i} = \frac{1}{c - a^2\left[E_{\psi_{j|\psi_i}} \max_{j\neq i}\{k_j(\psi_j)\}\right]^2}. \tag{86}$$

Under our assumption that the ψ_i's are independently distributed, the RHS of (86) does not depend on ψ_i and therefore so should the LHS. This suggests that $\frac{k_i(\psi_i)}{\psi_i} = \nu_i$, where ν_i is an deterministic endogenous variable, that is independent of ψ_i and $\psi_j,\ j \neq i$ but does depend on all parameters of the problem. That is, $k_i(\psi_i) = \psi_i\nu_i, \forall i \in \mathcal{I}$. Condition (86) may be rewritten as:

$$\nu_i = \frac{1}{c - a^2\left[E_{\psi_{j|\psi_i}} \max_{j\neq i}\{\nu_j\psi_j\}\right]^2},\ i \in \{\mathcal{I}\}. \tag{87}$$

Let us assume that the random variables ψ_j are Fréchet-distributed and conditionally independent, whose cumulative distribution is given by: $\exp\left[-\left(\frac{\psi - m_i}{\sigma_i}\right)^{-\chi}\right]$, where (m_i, σ_i, χ) are positive parameters, denoting the minimum, scale, and shape parameters, respectively. It follows that the cumulative distribution function of $\psi_i\nu_i$ is given by: $\exp\left[-\left(\frac{\kappa - m_i\sigma_i}{\nu_i\sigma_i}\right)^{-\chi}\right]$. The corresponding cumulative distribution function of $\max_{j\neq i}\{\sigma_j\nu_j\}$ is given $\Pi_{j\neq i}\left(\text{Prob}\left\{\psi_j\nu_j \leq \kappa\right\}\right)$. The expectation of $\max_{j\neq i}\{\psi_j\nu_j\}$ is obtained by integrating the density corresponding to the above cumulative distribution function. The expectation is a function of the ν_j's, and so is the RHS of (87). The unknown $(\nu_1, \ldots, \nu_i, \ldots, \nu_I)$ follow as solutions to the system of equations (87). The solutions for the networking efforts, the s_i's, follow from (85).

If the ψ_i's are identically distributed, $m_i = m, \sigma_i = \sigma$, then the expectation of $\max_{j\neq i}\{\psi_j\nu_j\}$ is readily obtained from the extreme order statistics theory and defines only one ratio, ν. The maximum of the realizations of a number of independently and identically Fréchet distributed random variables is also Fréchet distributed with scale parameter $I^{\frac{1}{\chi}}\sigma$. Its expectation is given in closed form by $m + I^{\frac{1}{I}}\nu\sigma\Gamma\left(1 - \frac{1}{\chi}\right)$, provided that $\chi > 1$. The unknown

ν satisfies

$$\nu = \frac{1}{c - a^2 \left[m + I^{\frac{1}{I}} \sigma \Gamma \left(1 - \frac{1}{\chi}\right)\right]^2 \nu^2}, \tag{88}$$

which is a cubic equation in ν. Depending upon parameter values, this equation may have either one or two feasible solutions, or none. Feasibility is conceptual similar to condition (15), with $\tilde{a}$ now defined as:

$$\tilde{a} \equiv a \left[m + I^{\frac{1}{\chi}} \sigma \Gamma \left(1 - \frac{1}{\chi}\right)\right]. \tag{89}$$

Q.E.D.

8.5 Proposition 5. Proof

It is easier to work with the scalar versions of Eq. (22) – (23):

$$k_{it} = \frac{1}{c} b_{\tau(i)} + \frac{a}{c} \sum_{j=1, j\neq i}^{I} g_{ij}(\mathbf{s}_{t-1}) k_{j,t-1}; \tag{90}$$

$$s_{it} = a\rho \sum_{j=1, j\neq i}^{I} k_{it+1} k_{jt} \frac{\partial g_{ij}(\mathbf{s}_t)}{\partial s_{it}}, \tag{91}$$

The proof of Part A is straightforward. Part B follows readily once we remove the time subscripts. Regarding Part C we work as follows. By linearizing system (22–23) in the standard fashion and by denoting by $\Delta x_{it} = x_{it} - x_i^*$ deviations from steady-state values, we have:

$$\Delta k_{it} = \frac{a}{c} \sum_{j=1, j\neq i}^{I} g_{ij}(\mathbf{s}^*) \Delta k_{j,t-1} + \frac{a}{c} \sum_{j=1, j\neq i}^{I} k_j^* \sum_{h=1}^{I} \frac{\partial g_{ij}}{\partial s_h}|_{\mathbf{s}^*} \Delta s_{h,t-1}. \tag{92}$$

$$\Delta s_{it} = a\rho k_i^* \sum_{j=1, j\neq i}^{I} k_j^* \sum_{h=1}^{I} \frac{\partial^2 g_{ij}}{\partial s_i \partial s_h}|_{\mathbf{s}^*} \Delta s_{ht} + a\rho k_i^* \sum_{j=1, j\neq i}^{I} \frac{\partial g_{ij}}{\partial s_i}|_{\mathbf{s}^*} \Delta k_{jt} + \left(a\rho \sum_{j=1, j\neq i}^{I} k_j^* \frac{\partial g_{ij}}{\partial s_i}|_{\mathbf{s}^*}\right) \Delta k_{it+1}, \tag{93}$$

where except for the time-subscripted variables, all others assume their steady-state values.

The asymptotic results invoked earlier allow us to simplify these conditions.First, we note that:

$$\frac{\partial g_{ij}}{\partial s_h} = -\frac{s_i s_j}{\left(\sum_{h=1}^{I} s_h\right)^2}, \quad h \neq i, j;$$

$$\frac{\partial g_{ij}}{\partial s_i} = \frac{s_j \sum_{h \neq i} s_h}{\left(\sum_{h=1}^{I} s_h\right)^2};$$
$$\frac{\partial g_{ij}}{\partial s_j} = \frac{s_i \sum_{h \neq i} s_h}{\left(\sum_{h=1}^{I} s_h\right)^2};$$

The terms $s_j \sum_{h \neq i} s_h$ tend to $I\varpi^2\vartheta^2\bar{\mathbf{b}}$, and the terms $\left(\sum_{h=1}^{I} s_h\right)^2$ tend to $\varpi\vartheta I^2(\bar{\mathbf{b}})^2$. Therefore, as $I \to \infty$, the derivatives above tend to zero and the second term in the rhs of (92) and of (93) vanish. Similarly, the first term in the rhs of (93) also vanishes. System (92–93) may now be written as follows, where we advance the time subscript for t in the first equation:

$$\Delta\mathbf{k}_{t+1} = \frac{a}{c}\mathbf{G}(\mathbf{s}^*)\Delta\mathbf{k}_t, \tag{94}$$

$$\Delta\mathbf{s}_t = \rho\tilde{a}\vartheta\Delta\mathbf{k}_{t+1}. \tag{95}$$

By using (94) in (95) we see that the changes in networking efforts, $\Delta\mathbf{s}_t$, are determined by the contemporaneous values of the changes in human capitals, the $\Delta\mathbf{k}_t$'s. That is:

$$\Delta\mathbf{s}_t = \frac{a\tilde{a}(\mathbf{b})\rho\vartheta}{c}\mathbf{G}(\mathbf{s}^*)\Delta\mathbf{k}_t. \tag{96}$$

The dynamic evolution of the human capitals is determined by (24), and therefore of the networking efforts as well through (96).

The properties of the matrix $\frac{a}{c}\mathbf{G}(\mathbf{s}^*)$ fully determines the dynamics, and its properties are in turn determined by those of the steady state solutions. We know from Cabrales *et al.* (2011) that the largest eigenvalue of $\mathbf{G}(\mathbf{s}^*)$ is equal to $\frac{\overline{x^2}(\mathbf{s}^*)}{\overline{x}(\mathbf{s}^*)}$ and corresponds to $\mathbf{s}$ as an eigenvector. Therefore, the condition

$$\frac{a}{c}\frac{\overline{x^2}(\mathbf{s})}{\overline{x}(\mathbf{s})} = \frac{1}{c}\varpi\vartheta\tilde{a} < 1$$

is sufficient for the stability of the solution of (94). In view of Proposition 1, this condition becomes:

$$\varpi^2 < c,$$

which is satisfied for both non-zero steady states.

For the stability of (96) it is required that

$$\frac{a\tilde{a}(\mathbf{b})\rho\vartheta}{c}\frac{\overline{x^2}(\mathbf{s}^*)}{\overline{x}(\mathbf{s}^*)} = \frac{\rho}{c}\varpi^3 < 1.$$

A sufficient condition for this to hold is that $\rho < \varpi^{-1} < 1$. Q.E.D.

8.6 Proposition 6. Proof

As indicated in the text, we assume that the social interactions matrix $\tilde{\mathbf{G}}_t = \tilde{\mathbf{G}}(\mathbf{\Phi}_t)$ is defined to include the diagonal terms too. We assume that the pairs $\left\{\tilde{\mathbf{G}}_t, \mathbf{\Psi}_t\right\}$ are independently and identically distributed elements of a stationary stochastic process with positive entries. Adopting as matrix norm $||\cdot||$ for $I \times I$ matrices the function $||m|| = \max_{|y|=1} |ym|$, where y denotes an I row vector, and m denotes an $I \times I$ matrix.[24] If

$$\mathcal{E} \ln^+ ||\tilde{\mathbf{G}}(\mathbf{\Phi}_1)|| < 0,$$

then

$$\mathrm{Lim}_1 = \lim \left(\ln ||\tilde{\mathbf{G}}(\mathbf{\Phi}_1) \cdots \tilde{\mathbf{G}}(\mathbf{\Phi}_t)||^{\frac{1}{t}}\right) \tag{97}$$

exists, is constant and finite w.p. 1. If we assume that the $\tilde{\mathbf{G}}$'s are such that $\mathrm{Lim}_1 < 0$, then $||\tilde{\mathbf{G}}(\mathbf{\Phi}_1) \cdots \tilde{\mathbf{G}}(\mathbf{\Phi}_t)||$ converges to 0 exponentially fast. If $|\Psi_1|^\kappa < \infty$ for some $\kappa > 0$, that is if the starting shock is not too large, with the norm $|\cdot|$ being defined as the Euclidian norm, then the series of the vectors of human capital

$$\mathbf{K} \equiv \sum_{t=1}^{\infty} \tilde{\mathbf{G}}(\mathbf{\Phi}_1) \cdots \tilde{\mathbf{G}}(\mathbf{\Phi}_{t-1}) \mathbf{\Psi}_t$$

converges w. p. 1, and the distribution of the solution $\tilde{\mathbf{k}}_t$ of (31) converges to that of $\mathbf{K}$, independently of $\tilde{\mathbf{k}}_0$. This is simply a rigorous way to establish the limit human capital vector.

In particular, from (97), if $\mathrm{Lim}_1 < 0$, then the norm of the product of t successive social interactions matrices, raised to the power of t^{-1}, is positive but less than 1. In that case, Kesten (1973) shows that the distribution of $\mathbf{K}$ can have a thick upper tail. That is, according to Kesten (1973), Theorem A, if in addition to the above conditions there exists a constant $\kappa_0 > 0$, for which

$$\mathcal{E}\left\{\frac{1}{I^{\frac{1}{2}}} \min_i \left(\sum_{j=1}^{I} \tilde{\mathbf{G}}_{1_{i,j}}\right)\right\}^{\kappa_0} \geq 1, \text{and } \mathcal{E}\left\{||\tilde{\mathbf{G}}_1||^{\kappa_0} \ln^+ ||\tilde{\mathbf{G}}_1||\right\} < \infty, \tag{98}$$

then there exists a $\kappa_1 \in (0, \kappa_0]$ such that

$$\lim_{v\to\infty} \mathrm{Prob}\left\{\max_{n\geq 0} |x\tilde{\mathbf{G}}_1 \cdots \tilde{\mathbf{G}}_n| > v\right\} \sim X(x) v^{-\kappa_1}, \tag{99}$$

where $0 \leq X(x) < \infty$, with $X(x) > 0$, where the (row) vector x belongs to the positive orthant of the unit sphere of $I\!R^I$, exists and is strictly positive. If, in addition, the components of Ψ_1 satisfy:

$$\text{Prob}\{\mathbf{\Psi}_1 = 0\} < 1, \ \text{Prob}\{\mathbf{\Psi}_1 \geq 0\} = 1, \ \mathcal{E}|\mathbf{\Psi}_1|^{\kappa_1} < \infty,$$

then for all elements x on the unit sphere in $I\!R^I$, then condition (33) follows. That is, the upper tail of the distribution of $x\mathbf{K}$,

$$\lim_{v\to\infty} v^{\kappa_1} \text{Prob}\{x\mathbf{K} \geq v\} \tag{100}$$

exists, is finite and for all elements x on the positive orthant of the unit sphere in $I\!R^I$ is strictly positive.

The intuition of condition (98) is that if there exists a positive constant κ_0, for which the expectation of the minimum row sum of the social interactions matrix raised to the power of κ_0, grows with the number of agents I faster than $\sqrt{I}$, roughly speaking, but does not grow too fast so as to blow up, then the contracting effect of the social interactions system does not send human capitals to zero, when the economy starts from an arbitrary initial condition, say when when all initial human capitals are uniformly distributed. The intuition of condition (97) is that the geometric mean of the limit of the sequence of norms of the social interactions matrix is positive but less than 1. Q.E.D.

8.7 Proposition 7. Proof

The decision problem for a member of generation t, born at time t, is to choose

$$\{k_{o,i,t+1}, k_{y,i,t+1}; s_{y,i,t,}, s_{o,i,t+1}\},$$

given $\{k_{y,i,t}, \mathbf{s}_{o,t}\}$. We express the first-order conditions by first defining the value functions $\mathcal{V}_i^{[t]}(k_{y,i,t}, \mathbf{s}_{o,t}), \mathcal{V}_i^{[t+1]}(k_{y,i,t+1}, \mathbf{s}_{o,t+1})$, associated with an individual's lifetime utility when he is young at t and when he is old at $t+1$, we have:

$$\mathcal{V}^{[t]}(k_{y,i,t}, \mathbf{s}_{o,t})$$

$$= \max_{\{k_{o,i,t+1}, k_{y,i,t+1}; s_{y,i,t,}, s_{o,i,t+1}\}} \Bigg\{ b_{y,i,t}k_{y,i,t} + a\sum_{j\neq i} g_{ij}(\mathbf{s}_{o,t})k_{y,i,t}k_{o,j,t} - \frac{1}{2}ck_{y,i,t}^2 - \frac{1}{2}s_{y,i,t}^2 - k_{o,i,t+1}$$

$$+\rho\left[b_{o,i,t+1}k_{o,i,t+1}+a\sum_{j\neq i}g_{ij}(\mathbf{s}_{y,t})k_{o,i,t+1}k_{y,j,t}-\frac{1}{2}ck_{o,i,t+1}^2-\frac{1}{2}s_{o,i,t+1}^2-k_{y,i,t+1}\right]+\rho\mathcal{V}_i^{[t+1]}(k_{y,i,t+1},\mathbf{s}_{o,t+1})\Bigg\}.$$

Correspondingly,

$$\mathcal{V}_i^{[t+1]}(k_{y,i,t+1},\mathbf{s}_{o,t+1})$$

$$=\max_{\{k_{o,i,t+2},k_{y,i,t+2};s_{y,i,t+1},,s_{o,i,t+2}\}}\Bigg\{b_{y,i,t+1}k_{y,i,t+1}+a\sum_{j\neq i}g_{ij}(\mathbf{s}_{o,t+1})k_{y,i,t+1}k_{o,j,t+1}-\frac{1}{2}ck_{y,i,t+1}^2-\frac{1}{2}s_{y,i,t+1}^2-k_{o,i,t+2}$$

$$+\rho\left[b_{o,i,t+2}k_{o,i,t+2}+a\sum_{j\neq i}g_{ij}(\mathbf{s}_{y,t+1})k_{o,i,t+2}k_{y,j,t+1}-\frac{1}{2}ck_{o,i,t+2}^2-\frac{1}{2}s_{o,i,t+2}^2-k_{y,i,t+2}\right]+\rho\mathcal{V}_i^{[t+2]}(k_{y,i,t+2},\mathbf{s}_{o,t+2})\Bigg\}$$

Parts A and B readily follow. The first-order conditions with respect to $(k_{o,i,t+1},s_{y,i,t,};k_{y,i,t+1},s_{o,i,t+1})$ are, respectively:

$$k_{o,i,t+1}=\frac{1}{c}b_{o,i,t+1}+\frac{a}{c}\sum_{j\neq i}g_{ij}(\mathbf{s}_{y,t})k_{y,j,t}-\frac{1}{c\rho};\tag{101}$$

$$s_{y,i,t}=\rho ak_{o,i,t+1}\sum_{j=1,j\neq i}^{I}\frac{\partial g_{ij}}{\partial s_{y,i,t}}(\mathbf{s}_{y,t})k_{y,j,t};\tag{102}$$

$$-\rho+\rho\frac{\partial\mathcal{V}_i^{[t+1]}}{\partial k_{y,i,t+1}}(k_{y,i,t+1},\mathbf{s}_{o,t+1})=0;$$

$$-\rho s_{o,i,t+1}+\rho\frac{\partial\mathcal{V}_i^{[t+1]}}{\partial s_{o,i,t+1}}(k_{y,i,t+1},\mathbf{s}_{o,t+1})=0.$$

Using the envelope property, the partial derivatives of the value function above,

$$\frac{\partial\mathcal{V}_i^{[t+1]}}{\partial k_{y,i,t+1}}(k_{y,i,t+1},\mathbf{s}_{o,t+1}),\ \frac{\partial\mathcal{V}^{[t+1]}}{\partial s_{o,i,t+1}}(k_{y,i,t+1},\mathbf{s}_{o,t+1})$$

are equal to the partial derivatives of the respective utility per period. That is, using the envelope property, the last two equations become:

$$k_{y,i,t+1}=\frac{1}{c}b_{y,i,t+1}+\frac{a}{c}\sum_{j\neq i}g_{ij}(\mathbf{s}_{o,t+1})k_{o,j,t+1}-\frac{1}{c\rho};\tag{103}$$

$$s_{o,i,t+1}=\rho ak_{y,i,t+1}\sum_{j=1,j\neq i}^{I}\frac{\partial g_{ij}}{\partial s_{o,i,t+1}}(\mathbf{s}_{o,t+1})k_{o,j,t+1};\tag{104}$$

We can summarize the first-order conditions for the **k**'s in matrix form as follows.

$$\mathbf{k}_{o,t+1}=\frac{1}{c}\mathbf{b}_{o,t+1}+\frac{a}{c}\mathbf{G}(\mathbf{s}_{y,t})\mathbf{k}_{y,t}-\frac{1}{c\rho}\mathbf{1};\tag{105}$$

$$\mathbf{k}_{y,t+1} = \frac{1}{c}\mathbf{b}_{y,t+1} + \frac{a}{c}\mathbf{G}(\mathbf{s}_{o,t+1})\mathbf{k}_{o,t+1} - \frac{1}{c\rho}\mathbf{1}, \tag{106}$$

where $\mathbf{1}$ is a $I-$ vector of 1's. From these we may obtain two single first-order difference equations: first in $\mathbf{k}_{y,t}$, by substituting for $\mathbf{k}_{o,t+1}$ from (105) in the rhs of (106), and then in $\mathbf{k}_{y,t}$, by substituting for $\mathbf{k}_{y,t}$ from (106) in the rhs of (105). That is, (38 – 39) in the main text follow, reproduced here as well for clarity:

$$\mathbf{k}_{y,t+1} = \frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_{y,t})\mathbf{G}(\mathbf{s}_{o,t+1})\mathbf{k}_{y,t} + \frac{1}{c}\mathbf{b}_{y,t+1} + \frac{a}{c^2}\mathbf{G}(\mathbf{s}_{o,t+1})\mathbf{b}_{o,t+1} - \frac{1}{c\rho}\left[\mathbf{I} + \frac{a}{c}\mathbf{G}(\mathbf{s}_{o,t+1})\right]\mathbf{1}. \tag{107}$$

$$\mathbf{k}_{o,t+1} = \frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_{o,t})\mathbf{G}(\mathbf{s}_{y,t})\mathbf{k}_{o,t} + \frac{1}{c}\mathbf{b}_{o,t+1} + \frac{a}{c^2}\mathbf{G}(\mathbf{s}_{y,t})\mathbf{b}_{y,t} - \frac{1}{c\rho}\left[\mathbf{I} + \frac{a}{c}\mathbf{G}(\mathbf{s}_{y,t})\right]\mathbf{1}. \tag{108}$$

Part C. Since the largest eigenvalue of $\mathbf{G}(\mathbf{s}_o)\mathbf{G}(\mathbf{s}_y)$ is bounded upwards by the product of the largest eigenvalues of $\mathbf{G}(\mathbf{s}_o)$ and $\mathbf{G}(\mathbf{s}_y)$ [Debreu and Herrstein (1953); Merikoski and Kumar (2006), Thm. 7, 154–155], the inverse exists, provided that the product of $\frac{a^2}{c^2}$ with the largest eigenvalues of $\mathbf{G}(\mathbf{s}_o)$ and of $\mathbf{G}(\mathbf{s}_y)$ is less than 1. A sufficient condition for this is that the products of $\frac{a}{c}$ and each of the largest eigenvalues of $\mathbf{G}(\mathbf{s}_o), \mathbf{G}(\mathbf{s}_y)$ are less than 1.

Part D. We follow the line of proof in Lemma 3, Cabrales et al. (2011), p. 353, we explore whether $\left[\mathbf{I} - \left(\frac{a}{c}\right)^2 \mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)\right]^{-1}$ may be written in close form. Writing out the generic element of the matrix product $\mathbf{G}(\mathbf{s}_y;\mathbf{s}_o)$ yields

$$\mathbf{G}(\mathbf{s}_y;\mathbf{s}_o)_{i,j} = \frac{\sum_\ell s_{y,\ell}s_{o,\ell}}{I\overline{x}(\mathbf{s}_o)}\,\frac{s_{y,i}s_{o,j}}{I\overline{x}(\mathbf{s}_y)}.$$

For the higher powers of $\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)$ we use the symmetry of each of the matrices $\mathbf{G}(\mathbf{s}_y), \mathbf{G}(\mathbf{s}_o)$ and the result in *ibid.* to write for the generic element of $\mathbf{G}(\mathbf{s}_y)$ (and similarly for $\mathbf{G}(\mathbf{s}_o)$) as follows:

$$[\mathbf{G}(\mathbf{s}_y)]^2_{i,j} = \frac{\overline{x^2}(\mathbf{s}_y)}{\overline{x}(\mathbf{s}_y)}[\mathbf{G}(\mathbf{s}_y)]_{i,j}.$$

Thus by trivial induction and provided that condition (42) in the main text holds, the power expansion for the above matrix converges and is given by (43) in the main text.

Q.E.D.

8.8 Proposition 8. Proof

Part A readily follows from the derivations in the main text and the following derivation, for the total effect of an increase in first period wealth on the transfer to the child. That is, from (38) and (41) we have:

$$\frac{d\ k_{y,i,t+1}}{d\ k_{y,i,t}} = \frac{\partial k_{y,i,t+1}}{\partial k_{y,i,t}} \left[1 + \rho a \sum_{j=1,j\neq i}^{I} \frac{\partial g_{ij}}{\partial s_{o,i,t+1}} (\mathbf{s}_{o,t+1}) k_{o,j,t+1} \frac{\partial s_{o,i,t+1}}{\partial k_{y,i,t+1}} \right],$$

where the partial derivative of $\mathbf{k}_{y,t+1}$ with respect to $s_{o,i,t+1}$ is given by:

$$\frac{a^2}{c^2} \mathbf{G}(\mathbf{s}_{y,t}) \frac{\partial}{\partial s_{o,i,t+1}} \mathbf{G}(\mathbf{s}_{o,t+1}) \mathbf{k}_{y,t} + \frac{\partial}{\partial s_{o,i,t+1}} \mathbf{G}(\mathbf{s}_{o,t+1}) \left[\frac{a}{c^2} \mathbf{b}_{o,t+1} - \frac{a}{\rho c^2} \mathbf{1} \right],$$

with

$$\frac{\partial}{\partial s_{o,i,t+1}} \mathbf{G}(\mathbf{s}_{o,t+1}) = \begin{bmatrix} 0 & 0 & \dots & \frac{s_{o,1,t+1}}{\sum_{j\neq 1} s_{o,1,t+1}} & \dots & 0 \\ \frac{s_{o,1,t+1}}{\sum_{j\neq i} s_{o,j,t+1}} & \frac{s_{o,2,t+1}}{\sum_{j\neq i} s_{o,j,t+1}} & \dots & 0 & \dots & \frac{s_{o,I,t+1}}{\sum_{j\neq i} s_{o,j,t+1}} \\ 0 & 0 & \dots & \frac{s_{o,I,t+1}}{\sum_{j\neq I} s_{o,j,t+1}} & \dots & 0 \end{bmatrix},$$

Part B follows by inspection of (46), and provided that the sufficient conditions for the positivity of $(\mathbf{k}_{y,t}, \mathbf{k}_{o,t})$ in Part B, Proposition 7, hold. Q.E.D.

8.9 Proposition 9. Proof

Part A. By applying equations (105), (102), (106), and (104) we have:

$$ck_{o,i} = b^*_{o,i} + as_{y,i} \sum_{j\neq i} \frac{s_{y,j} k_{y,j}}{\sum_i s_{y,i}}; \tag{109}$$

$$s_{y,i} = \rho a k_{o,i} \sum_{j=1,j\neq i}^{I} \frac{s_{y,j} k_{y,j}}{\sum_i s_{y,i}}; \tag{110}$$

$$ck_{y,i} = b^*_{y,i} + as_{o,i} \sum_{j\neq i} \frac{s_{o,j} k_{o,j}}{\sum_i s_{o,i}}; \tag{111}$$

$$s_{o,i} = \rho a k_{y,i} \sum_{j=1,j\neq i}^{I} \frac{s_{o,j} k_{o,j}}{\sum_i s_{o,i}}. \tag{112}$$

Note that the auxiliary variables, ψ_y, ψ_o, defined in the main text do not depend on i. From (109) and (110), and (112) and (112), we have:

$$\rho k_{o,i}(ck_{o,i} - b^*_{o,i}) = s^2_{y,i} = \rho^2 a^2 \psi_y^2 k^2_{o,i};$$

$$\rho k_{y,i}(ck_{y,i} - b^*_{y,i}) = s^2_{o,i} = \rho^2 a^2 \psi_o^2 k^2_{y,i}.$$

We may thus solve for $k_{y,i}, k_{o,i}$, and then by using the definitions of ψ_y, ψ_o, for $s_{y,i}, s_{o,i}$, we obtain solutions for $k_{y,i}, k_{o,i}$ and $s_{y,i}, s_{o,i}$ in terms of (ψ_y, ψ_o) as in (48–49) in the main text. Finally, by substituting back into the definitions of ψ_y, ψ_o, we obtain obtain third-degree equations in ψ_y, ψ_o, (50–51).

Part B. Equations (50–51) have at most two solutions in (ψ_y, ψ_o), provided that

$$\frac{\mathbf{b}^*_y \cdot \mathbf{b}^*_o}{I\overline{x}(\mathbf{b}^*_y)} < \frac{c}{a}\left(\frac{c}{\rho}\right)^{\frac{1}{2}}; \frac{\mathbf{b}^*_y \cdot \mathbf{b}^*_o}{I\overline{x}(\mathbf{b}^*_o)} < \frac{c}{a}\left(\frac{c}{\rho}\right)^{\frac{1}{2}}.$$

Q.E.D.

8.10 Proposition 10. Proof

Part A. Transforming the individual's decision problem in the obvious way allows us to derive first order conditions, the stochastic counterpart of (38)–39). They are as follows:

$$k_{y,i,t+1} = \frac{1}{c}\mathcal{E}[b_{y,i,t+1}|b_{y,i,t};t] + \frac{a}{c}\sum_{j\neq i} g_{ij}(\mathbf{s}_o)\mathcal{E}[k_{o,j,t+1}|i,t] - \frac{1}{c\rho}; \tag{113}$$

$$k_{o,i,t+1} = \frac{1}{c}\mathcal{E}[b_{o,i,t+1}|b_{y,i,t};t] + \frac{a}{c}\sum_{j\neq i} g_{ij}(\mathbf{s}_y)\mathcal{E}[k_{y,j,t}|i,t] - \frac{1}{c\rho}. \tag{114}$$

These conditions may be rewritten readily as in the main text.

For Part B, from the stochastic assumptions we have that:

$$\mathcal{E}[b_{o,i,t+1}|b_{y,i,t}] = m_{o,i} + \frac{\sigma_o}{\sigma_y}\rho_o(b_{y,i,t} - b_{m,y,i}); \mathcal{E}[b_{y,i,t+1}|b_{y,i,t}] = (1-\rho_b)b_{m,y,i} + \rho_b b_{y,i,t}.$$

These expressions are used to write (53)–(54) by defining $\mathbf{G}_{\mathrm{adj,y}}(\mathbf{s}_o), \mathbf{G}_{\mathrm{adj,o}}(\mathbf{s}_y)$, in the form of (55)–(56).

Parts C and D readily by Proposition 4.1 of Bertsekas (1995): $\Delta\mathbf{k}_{y,t} = \mathbf{k}_{y,t} - \mathbf{k}^*_y$ has a multivariate normal limit distribution with mean $\mathbf{0}$ and variance covariance matrix Σ_∞ that satisfies (59) in the main text. The explicit solutions for $\boldsymbol{\Sigma}_{y,\infty}, \boldsymbol{\Sigma}_{o,\infty}$ follow by iterating (59), if the matrix $\frac{a^2}{c^2}\mathbf{G}(\mathbf{s}_y)\mathbf{G}(\mathbf{s}_o)$ is stable.

For Part E, consider the discrete random variable J taking values in $\{1, 2, \ldots, I-1, I\}$, with equal probabilities I^{-1}, and define the random vector $D = (d_1, d_2, \ldots, d_i, \ldots, d_I)$, with $d_i = 1,$ iff $i = J; d_i = 0,$ iff $i \neq J$. We assume that the shocks introduced in the main part of Proposition 10 are statistically independent of the random index J and the corresponding dummy random vector D. Finally, consider the univariate random variable $\mathcal{Z}_t$ that consists of randomly selecting on element the human capital vector, that is "anonymizing" this vector:

$$\mathcal{Z}_t = D^{\mathrm{T}}\Delta\mathbf{k}_{y,t} = \sum_i D_i \Delta k_{y,i,t}.$$

In this representation, one and only one of the D_i binary random variables will take the value 1 and all the others will be 0, so $\mathcal{Z}_t = \Delta k_{y,i,t}$ with equal probability I^{-1}. Since the D_i's are fully independent of the $\Delta k_{y,i,t}$'s, and each D_i takes the value 1 with equal probability I^{-1}, the expressions in (63) in the main text follow. The full probability density and distribution functions of $\mathcal{Z}_t$ follow directly from its definition. It is a mixture of univariate normal distributions. It is important to note that since only one of the $\Delta k_{y,i,t}$ is realized at any one time, the covariance/correlation structure between the $\Delta k_{y,i,t}$ is irrelevant. Only the individual variances matter. Still, the results reported in (63) reflect the social structure and, in addition, ensure a much richer outcome, as the cross-sectional distribution might no longer be unimodal.

Similar derivations readily follow for $k_{o,i,t}$ and the joint distribution of $(k_{y,i,t}, k_{y,i,t+1})$, the latter being the stochastic counterpart of Social Effects in Intergenerational Wealth Transfer Elasticities, discussed in section 5.1.2 of the main text. This involves deriving an expression for the covariance of $(\Delta k_{y,i,t+1}, \Delta k_{y,i,t})$ as:

$$\mathrm{Covar}(\Delta k_{y,i,t+1}, \Delta k_{y,i,t}) = \mathcal{E}[k_{y,i,t+1}k_{y,i,t}] - (k^*_{y,i})^2,$$

which involves elementary but tedious derivations. Q.E.D.

8.11 Proposition 11. Value functions

$$\mathcal{V}^{[t]}(k_{y,i,t}, \mathbf{s}_{o,t}; a_{i,t})$$

$$= \max_{\mathcal{E}_{a_{i,t+1}}\{k_{o,i,t+1}, k_{y,i,t+1}; s_{y,i,t,}, s_{o,i,t+1}\}} \mathcal{E}_{a_{i,t+1}} \Bigg\{ b_{y,i,t}k_{y,i,t} + a_{i,t}\sum_{j\neq i} g_{ij}(\mathbf{s}_{o,t})k_{y,i,t}k_{o,j,t} - \frac{1}{2}ck_{y,i,t}^2 - \frac{1}{2}s_{y,i,t}^2 - k_{o,i,t+1}$$

$$+\rho\left[b_{o,i,t+1}k_{o,i,t+1} + a\sum_{j\neq i} g_{ij}(\mathbf{s}_{y,t})k_{o,i,t+1}k_{y,j,t} - \frac{1}{2}ck_{o,i,t+1}^2 - \frac{1}{2}s_{o,i,t+1}^2 - k_{y,i,t+1}\right] + \rho\mathcal{V}_i^{[t+1]}(k_{y,i,t+1}, \mathbf{s}_{o,t+1}; a_{i,t+1})\Bigg\};$$

$$\mathcal{V}_i^{[t+1]}(k_{y,i,t+1}, \mathbf{s}_{o,t+1}; a_{i,t+1})$$

$$= \max_{\{k_{o,i,t+2}, k_{y,i,t+2}; s_{y,i,t+1,}, s_{o,i,t+2}\}} \mathcal{E}_{a_{i,t+2}} \Bigg\{ b_{y,i,t+1}k_{y,i,t+1} + a_{i,t+1}\sum_{j\neq i} g_{ij}(\mathbf{s}_{o,t+1})k_{y,i,t+1}k_{o,j,t+1} - \frac{1}{2}ck_{y,i,t+1}^2 - \frac{1}{2}s_{y,i,t+1}^2 - k_{o,i,t+2}$$

$$+\rho\left[b_{o,i,t+2}k_{o,i,t+2} + a_{i,t+1}\sum_{j\neq i} g_{ij}(\mathbf{s}_{y,t+1})k_{o,i,t+2}k_{y,j,t+1} - \frac{1}{2}ck_{o,i,t+2}^2 - \frac{1}{2}s_{o,i,t+2}^2 - k_{y,i,t+2}\right] + \rho\mathcal{V}_i^{[t+2]}(k_{y,i,t+2}, \mathbf{s}_{o,t+2}; a_{i,t+2})\Bigg\}$$

8.12 Proposition 12. Proof

An individual born at t takes cognitive skills and human capital as given, $(b_{y,i,t}, k_{y,i,t})$, and benefits from the networking efforts of the parents' generation, $\mathbf{s}_{o,t-1}$, who are in the third subperiod of their lives when she is born. She chooses at time t the second subperiod human capital and the first subperiod transfer received by the child at time $t+2$, respectively $\{k_{o,i,t+1}, k_{y,i,t+2}\}$; and the first and second subperiod networking efforts, $\{s_{y,i,t,}, s_{o,i,t+1}\}$, respectively. These benefit herself in the second subperiod of her life, and benefit her child too, when the child is in her first subperiod of her life and she herself in her third subperiod of her life. For analytical convenience, I assume that the adjustment costs for decisions $\{s_{y,i,t,}, k_{o,i,t+1}\}$, are both incurred in period t. The optimization problem implies that the cognitive skills, $b_{y,i,t+2}$, of the individual's child and the transfer she receives when she becomes an adult, $k_{y,i,t+2}$, are determined simultaneously. The definition of the value function for the problem now changes to:

$$\mathcal{V}^{[t]}(k_{y,i,t}, \mathbf{s}_{o,t-1}) = \max_{\{k_{o,i,t+1}, k_{y,i,t+2}; \iota_{c1,t}, \iota_{c2,t+1}; s_{y,i,t,}, s_{o,i,t+1}\}} \Big\{\rho^2\mathcal{V}^{[t+2]}(k_{y,i,t+2}, \mathbf{s}_{o,t+1})$$

$$+b_{y,i,t}k_{y,i,t} + a\sum_{j\neq i} g_{ij}(\mathbf{s}_{o,t-1})k_{y,i,t}k_{o,j,t} - \frac{1}{2}ck_{y,i,t}^2 - \frac{1}{2}s_{y,i,t}^2 - k_{o,i,t+1} - \iota_{c1,t} - \frac{1}{2}\gamma_1\iota_{c1,t}^2 +$$

$$\rho\left[b_{o,i,t+1}k_{o,i,t+1} + a\sum_{j\neq i} g_{ij}(\mathbf{s}_{y,t})k_{o,i,t+1}k_{y,j,t} - \frac{1}{2}ck_{o,i,t+1}^2 - \frac{1}{2}s_{o,i,t+1}^2 - k_{y,i,t+2} - \iota_{c1,t+1} - \frac{1}{2}\gamma_1\iota_{c1,t+1}^2\right]\Bigg\}.$$

The first order conditions for $\iota_{1,t}, \iota_{2,t+1}$ are:

$$-1 - \gamma_1\iota_{c1,t} + \rho^2\frac{\partial\mathcal{V}^{[t+2]}(k_{y,i,t+2}, \mathbf{s}_{o,t+1})}{\partial b_{y,i,t+2}}\left[\frac{\partial b_{y,i,t+2}}{\partial\iota_{c1,t}} + \frac{\partial b_{o,i,t+3}}{\partial\iota_{c1,t}}\right] = 0.$$

$$-\rho[1 - \gamma_2\iota_{c2,t+1}] + \rho^2\frac{\partial\mathcal{V}^{[t+2]}(k_{y,i,t+2}, \mathbf{s}_{o,t+1})}{\partial b_{y,i,t+2}}\left[\frac{\partial b_{y,i,t+2}}{\partial\iota_{c2,t+1}} + \frac{\partial b_{o,i,t+3}}{\partial\iota_{c2,t+1}}\right] = 0.$$

Using the envelope property we rewrite the partial derivation of the value function above and get:

$$-1 - \gamma_1\iota_{1,t} + \rho^2\beta_1\left[k_{y,i,t+2} + \rho k_{o,i,t+3}\right] = 0.$$

$$-1 - \gamma_2\iota_{2,t+1} + \rho\beta_2\left[k_{y,i,t+2} + \rho k_{o,i,t+3}\right] = 0.$$

Solving for $\iota_{1,t}, \iota_{2,t+1}$ yields:

$$\iota_{1,t} = \frac{1}{\gamma_1}(\rho^2\beta_1[k_{y,i,t+2} + \rho k_{o,i,t+3}] - 1); \iota_{2,t+1} = \frac{1}{\gamma_2}(\rho\beta_2[k_{y,i,t+2} + \rho k_{o,i,t+3}] - 1).$$

This in turn yields condition (68) in the main text:

$$b_{y,i,t+2} = b_{o,i,t+3} = \beta_0 b_{y,i,t} + \rho\rho_\beta[k_{y,i,t+2} + \rho k_{o,i,t+3}] - \rho_\beta, \tag{115}$$

where the auxiliary parameter ρ_β is defined as $\rho_\beta \equiv \left(\rho\frac{\beta_1}{\gamma_1} + \frac{\beta_2}{\gamma_2}\right)$. For some of the analysis below we assume that $b_{y,i,t}$ is constant, so that cognitive skills do not necessarily steadily increase. Of course, such a figure could be incorporated.

It follows that the first-order condition for $k_{y,i,t+2}$ must reflect the influence that decision has, as implied by the optimization problem, on $b_{y,i,t+2}$. Since $b_{y,i,t+2} = b_{o,i,t+3}$ the utility per period from the last two subperiods of the child's lifetime contribute to the first-order conditions. The first order conditions are:

$$-\rho + \rho^2\frac{\partial\mathcal{V}^{[t+2]}(k_{y,i,t+2}, \mathbf{s}_{o,t+1})}{\partial k_{y,i,t+2}} + \rho^2\frac{\partial\mathcal{V}^{[t+2]}(k_{y,i,t+2}, \mathbf{s}_{o,t+1})}{\partial b_{y,i,t+2}}\frac{\partial b_{y,i,t+2}}{\partial k_{y,i,t+2}} = 0.$$

After using the envelope property and (115), this yields the following:

$$-1 + \rho\left[b_{y,i,t+2} + a\sum_{j\neq i} g_{ij}(\mathbf{s}_{o,t+1})k_{o,j,t+2} - ck_{y,i,t+2}\right] + \rho^2\rho_\beta k_{y,i,t+2} + \rho^3\rho_\beta k_{o,i,t+3} = 0.$$

This condition is rewritten as:

$$k_{y,i,t+2} = \frac{1}{c_{cs}} b_{y,i,t+2} + \frac{a}{c_{cs}} \sum_{j \neq i} g_{ij}(\mathbf{s}_{o,t+1}) k_{o,j,t+2} + \frac{\rho^2}{c_{cs}} \rho_\beta k_{o,i,t+3} - \frac{1}{\rho c_{cs}}, \tag{116}$$

where the auxiliary variable c_{cs} is defined as: $c_{cs} \equiv c - \rho\rho_\beta$. This condition may be rewritten by using (115) to eliminate $b_{y,i,t+2}$ by expressing it in terms of $(k_{y,i,t+2}, k_{o,i,t+3})$.

In addition, the first-order conditions for $k_{o,i,t+1}, s_{y,i,t,}, s_{o,i,t+1}$ are as follows:

$$k_{o,i,t+1} = \frac{1}{c} b_{o,i,t+1} + \frac{a}{c} \sum_{j \neq i} g_{ij}(\mathbf{s}_{y,t}) k_{y,j,t} - \frac{1}{c\rho}. \tag{117}$$

$$s_{y,i,t} = \rho a k_{o,i,t+1} \sum_{j=1, j \neq i}^{I} \frac{\partial g_{ij}(\mathbf{s}_{y,t})}{\partial s_{y,i,t}} k_{y,j,t}; \tag{118}$$

$$s_{o,i,t+1} = \rho a k_{y,i,t+1} \sum_{j=1, j \neq i}^{I} \frac{\partial g_{ij}(\mathbf{s}_{o,t+1})}{\partial s_{o,i,t+1}} k_{o,j,t+1}. \tag{119}$$

Conditions (118) and (119) are similar, respectively, to (40) and (41) and thus may be manipulated at the steady state in like manner to the steady state analysis in section 5.1.4 above. It is more convenient to write Eq. (117) by advancing the time subscript as follows:

$$k_{o,i,t+3} = \frac{1}{c} b_{o,i,t+3} + \frac{a}{c} \sum_{j \neq i} g_{ij}(\mathbf{s}_{y,t+2}) k_{y,j,t+2} - \frac{1}{c\rho}. \tag{120}$$

By using (68) to write for $b_{o,i,t+3}$ in terms of its solution in terms of $(k_{y,i,t+2}, k_{o,i,t+3})$ and rewriting the conditions for $(k_{y,i,t+2}, k_{o,i,t+3})$ in matrix form, we have:

$$\mathbf{k}_{o,t+3} = \frac{\beta_0}{\rho^* c} \mathbf{b} - \frac{\rho_\beta}{\rho^*} \mathbf{i} + \left[\frac{\rho\rho_\beta}{\rho^* c} \mathbf{I} + \frac{a}{\rho^* c} \mathbf{G}(\mathbf{s}_{y,t+2}) \right] \mathbf{k}_{y,t+2}, \tag{121}$$

where $\rho^* \equiv 1 - \frac{\rho^2 \rho_\beta}{c}$.

$$\mathbf{k}_{y,t+2} = \frac{\beta_0}{\tilde{\rho} c_{cs}} \mathbf{b} - \frac{1}{\tilde{\rho}\rho c_{cs}} \mathbf{i} + \frac{a}{\tilde{\rho} c_{cs}} \mathbf{G}(\mathbf{s}_{o,t+2}) \mathbf{k}_{o,t+2} + \frac{\rho^2 \rho_\beta}{c_{cs}} \mathbf{k}_{o,t+3}, \tag{122}$$

where $\tilde{\rho} \equiv 1 - \frac{\rho\rho_\beta}{c_{cs}}$. However, by substituting from (121) for $\mathbf{k}_{o,t+3}$ in the rhs of (122), we have:

$$\begin{aligned} &\left[\left(1 - \frac{\rho^3 \rho_\beta^2}{\rho^* \tilde{\rho} c c_{cs}} \right) \mathbf{I} - \frac{a \rho^2 \rho_\beta}{\rho^* \tilde{\rho} c c_{cs}} \mathbf{G}(\mathbf{s}_{y,t+2}) \right] \mathbf{k}_{y,t+2} \\ &= \beta_0 \left[\frac{\rho^2 \rho_\beta}{\tilde{\rho} \rho^* c c_{cs}} + \frac{1}{\tilde{\rho} c_{cs}} \right] \mathbf{b} - \left[\frac{1}{\tilde{\rho}\rho c_{cs}} + \frac{\rho^2 \rho_\beta^2}{\tilde{\rho}\rho^* c_{cs}} \right] \mathbf{i} + \frac{a}{\tilde{\rho} c_{cs}} \mathbf{G}(\mathbf{s}_{o,t+2}) \mathbf{k}_{o,t+2}. \end{aligned}$$

By dividing through by $1-\frac{\rho^3\rho_\beta^2}{\rho^*\tilde{\rho}cc_{cs}}$ and denoting

$$\hat{a} \equiv \frac{a\rho^2\rho_\beta}{\rho^*\tilde{\rho}c_{cs}}\left(1-\frac{\rho^3\rho_\beta^2}{\rho^*\tilde{\rho}cc_{cs}}\right)^{-1},$$

we may solve the previous equation with respect to $\mathbf{k}_{y,t+2}$ as follows:

$$\mathbf{k}_{y,t+2} = \left[\mathbf{I}-\frac{\hat{a}}{c}\mathbf{G}(\mathbf{s}_{y,t+2})\right]^{-1}\left[\mathbf{b}'_{\text{eff}}+\frac{a}{\tilde{\rho}c_{cs}}\mathbf{G}(\mathbf{s}_{o,t+2})\mathbf{k}_{o,t+2}\right],$$

where $\mathbf{b}'_{\text{eff}}$ is the resulting new constant. By substituting into the rhs of (121), we obtain a single first-order linear difference system in $\mathbf{k}_{o,t+2}$:

$$\mathbf{k}_{o,t+3} = \mathbf{b}_{\text{eff}}+\frac{a}{\rho^*c}\mathbf{G}(\mathbf{s}_{y,t+2})\left[\mathbf{I}-\frac{\hat{a}}{c}\mathbf{G}(\mathbf{s}_{y,t+2})\right]^{-1}\frac{a}{\tilde{\rho}c_{cs}}\mathbf{G}(\mathbf{s}_{o,t+2})\mathbf{k}_{o,t+2}, \tag{123}$$

where $\mathbf{b}_{\text{eff}}$ denotes the resulting constant. Thus, this equation depends on both networking efforts by the young and the old in two successive periods, $\mathbf{G}(\mathbf{s}_{y,t+2}), \mathbf{G}(\mathbf{s}_{o,t+2})$.

In a notable difference from the previous model, we now see a key new role for the social networking that individuals avail of when young. The product $\mathbf{G}(\mathbf{s}_{y,t+2})\mathbf{G}(\mathbf{s}_{o,t+2})$ is adjusted by $\left[\mathbf{I}-\frac{\hat{a}}{c}\mathbf{G}(\mathbf{s}_{y,t+2})\right]^{-1}$. Intuitively, this effect acts to reinforce the effects of social networking when young. This readily follows from (121) and (121) above. Feedbacks are generated due to the investment in cognitive skills. Mathematical results invoked upon earlier can still be used to determine the stability of (123). That is, $\left[\mathbf{I}-\frac{\hat{a}}{c}\mathbf{G}(\mathbf{s}_{y,2})\right]^{-1}$ admits a simple expression, following steps similar to those employed above, provided that the maximal eigenvalue of $\frac{\hat{a}}{c}\mathbf{G}(\mathbf{s}_{y,2})$ is less than 1, that is:

$$\frac{\hat{a}}{c}\frac{\overline{x^2}(\mathbf{s}_{y,2})}{\overline{x}(\mathbf{s}_{y,2})} < 1.$$

Thus:

$$\left[\mathbf{I}-\frac{\hat{a}}{c}\mathbf{G}(\mathbf{s}_{y,2})\right]^{-1} = \mathbf{I}+\frac{\hat{a}}{c}\frac{\overline{x}(\mathbf{s}_{y,2})}{\overline{x}(\mathbf{s}_{y,2})-\frac{\hat{a}}{c}\overline{x^2}(\mathbf{s}_{y,2})}\mathbf{G}(\mathbf{s}_{y,2}).$$

Thus, the stability of (123) rests on the spectral properties of

$$\frac{a}{\rho^*c}\,\frac{a}{\tilde{\rho}c_{cs}}\mathbf{G}(\mathbf{s}_{y,2})\mathbf{G}(\mathbf{s}_{o,2})+\frac{a}{\rho^*c}\,\frac{\hat{a}}{c}\,\frac{a}{\tilde{\rho}c_{cs}}\frac{\overline{x}(\mathbf{s}_{y,2})}{\overline{x}(\mathbf{s}_{y,2})-\frac{\hat{a}}{c}\overline{x^2}(\mathbf{s}_{y,2})}\mathbf{G}(\mathbf{s}_{y,2})^2\mathbf{G}(\mathbf{s}_{o,2}).$$

By Theorem 1, Merikoski and Kumar (2004), 151–152, the maximal eigenvalue of the sum of two real symmetric (Hermitian) matrices is bounded upwards by the sum of the maximal eigenvalues of the respective matrices. Thus, a condition for the stability of (123) readily follows and involves $(\mathbf{s}_{y,2}, \mathbf{s}_{o,2})$ along with the other parameters of the model. Q.E.D.